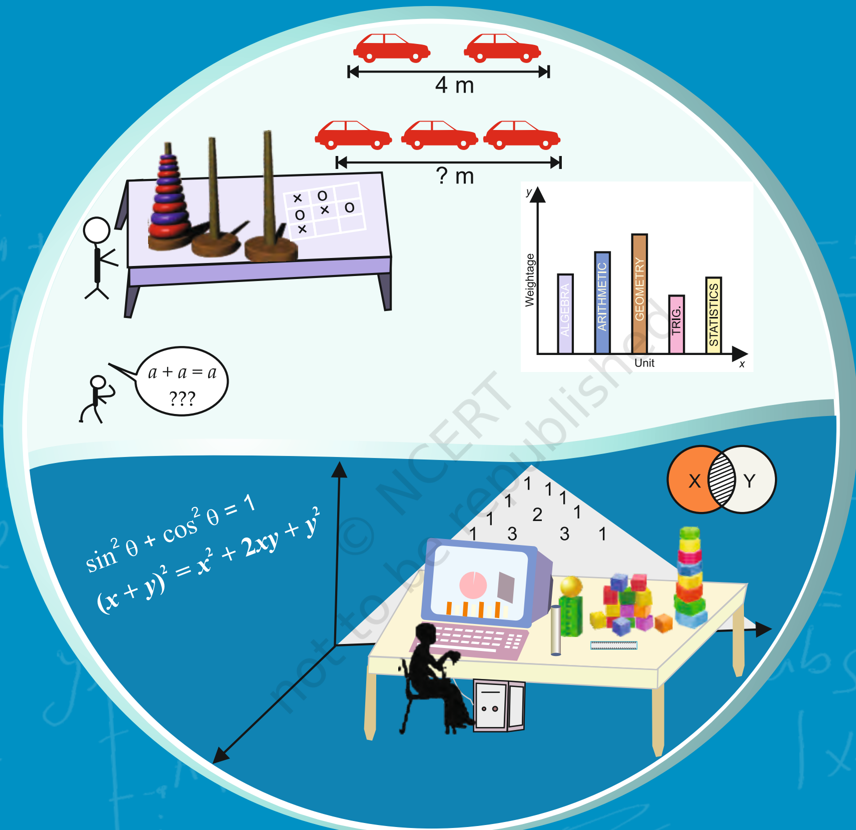
PEDAGOGY OF MATHEMATICS

Textbook for Two-Year B.Ed. Course



PEDAGOGY OF MATHEMATICS

Textbook for Two-Year B.Ed. Course

विद्यया ऽमृतमश्नुते

एन सी ई आर टी
NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

ISBN 978-93-5007-189-2

First Edition
February 2012 Magha 1933

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OFFICES OF THE PUBLICATION DEPARTMENT, NCERT

NCERT Campus
Sri Aurobindo Marg
New Delhi 110 016 **Phone : 011-26562708**

108, 100 Feet Road
Hosdakere Halli Extension
Banashankari III Stage
Bangluru 560 085 **Phone : 080-26725740**

Navjivan Trust Building
P.O.Navjivan
Ahmedabad 380 014 **Phone : 079-27541446**

CWC Campus
Opp. Dhankal Bus Stop
Panihati
Kolkata 700 114 **Phone : 033-25530454**

CWC Complex
Maligaon
Guwahati 781 021 **Phone : 0361-2674869**

Publication Team

Head, Publication Department	:	*Ashok Srivastava*
Chief Production Officer	:	*Shiv Kumar*
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Chief Business Manager	:	*Gautam Ganguly*
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Cover and Illusrations

Md. Iqbal Hassan

Printed on 80 GSM paper

Published at the Publication Department by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110 016 and printed at.....?

FOREWORD

The Position Paper on *Teacher Education for Curriculum Renewal* of NCERT observes that the exercise of revising school curriculum with the aim to revitalise school education cannot be achieved without addressing the need for creating reflective teacher practitioners. It envisions that the learning inputs in new teacher education programmes will be predominantly learner oriented as it would provide for variety in learning exposures, accommodate differential learning, encourage divergence, reflection and insightful treatment of a learning situation, and also provide for critical examination of disturbing social conditions of learners, larger issues of social disparity, inequity, gender divide and field specific administration and organisational anomalies – all of which contribute to each teacher evolving one's own conviction about teaching as a profession and a professional commitment. In this context the National Council of Educational Research and Training (NCERT) has developed syllabi for teacher education programme that signify the attempt to implement the above ideas. Based on the syllabi all the concerned departments have initiated the development of textbooks to support the student teachers. In the series the Department of Education in Science and Mathematics has prepared textbooks entitled Textbook on Pedagogy of Science (in two volumes on Physical Science and Biological Science) and Pedagogy of Mathematics. We hope that these books will serve the purpose of teacher education programme that can facilitate them in a child centered system of education.

The success of this effort would be possible if freedom and flexibility are given to teacher educators, student teachers and teachers at the school level in their teaching learning endeavours. Teachers need to recognise that every child learns in her/his own unique way. Therefore, every teacher has to find her/his own way of engaging the learners in the learning process. Teaching learning of science and mathematics should be closely intertwined with the contents and pedagogy of science and mathematics, respectively. Involving learners in the process of inquiry and investigation helps the teachers to gain a better insight into the nature of science and mathematics and purpose of science and mathematics education. We hope the textbooks will serve as a guide to teacher educators and student teachers in enhancing their professional competencies and motivating learners to learn science and mathematics as a process of investigation and to solve day-to-day life problems in a socially responsible manner as a global citizen.

The National Council of Education Research and Training appreciates the hard work done by the textbook development committees constituted for these textbooks. Several teachers also contributed to the development of the textbook. We are grateful to their Heads of Department and Principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, materials and personnel.

I sincerely acknowledge and appreciate the hard work done by Dr. Shashi Prabha, Prof. B.K. Tripathi and Dr. R.P. Maurya, member-coordinators, DESM and faculty members of NCERT who contributed to the development of the textbooks. I would also like to acknowledge the efforts of Prof. Hukum Singh, Head, Department of Education in Science and Mathematics (DESM) for his keen interest and continuous support. As an organisation committed to systemic reform and continuous improvement in the quality of its products and teacher education programme NCERT welcomes comments and suggestions which will enable us to undertake further refinement.

Director
National Council of Educational
Research and Training

New Delhi
April, 2011

PREFACE

National Curriculum Framework (NCF) – 2005 brought in a remarkable change in the field of school education when it strongly advocated for a child-centric education system. Such a change in curricular vision needed to be supported and sustained with systemic reforms of structures and institutions. The RIEs are expected to act as role models in the field of Teacher Education and to continually provide renewed conceptual orientation through teacher education programmes. The exercise of reformulation of teacher education programmes at the Regional Institutes of Education of the NCERT is part of a series of efforts initiated by the Council towards systemic reforms to strengthen school education.

The National Focus Group on Teacher Education formed during NCF-2005 exercise strongly recommended redesigning of teacher education programme to respond to the current changes in the school curriculum in the State and regional context. It further recommended to undertake a nation-wide review of these programmes.

Existing teacher education programmes are not sufficient to accommodate the emerging ideas in content and pedagogy as well as the issue of linkages between school and society. The purpose of such programmes is to understand school subjects and their pedagogic study in the concrete context of the school and the learner by establishing linkages among learner, context, subject discipline and pedagogical approach.

The key departure of pedagogical courses from conventional teacher education programmes shifted the focus from pure disciplinary knowledge and methodology to learner and his context. For instance, now a pedagogy course on mathematics would focus on understanding the nature of children's mathematical thinking as much through theory as through direct observation of children's thinking and learning process, the language of mathematics and engagement with children's learning in specific areas.

The nature of any discipline has significant impact on its pedagogy. One can state the nature of mathematics as having: Abstraction, Symbolic methods, Conditional reasoning, Rigour, High (and even unexpected) applicability to the real world and extremely long Historical development. The proposed pedagogical shift will help the children to understand and appreciate this nature of mathematics in a better way.

Teacher and student engagement is critical in a classroom. What children learn out of school, and bring to school is important to further enhance the learning processes by the teacher. This is all the more true for children from underprivileged backgrounds, especially girls, as the world they inhabit and their realities are under–represented in school knowledge. Participatory teaching and learning, emotion and experience need to have definite and important place in the classroom. Though classroom participation is a powerful strategy, it loses its pedagogic edge when it is ritualised, or when it merely becomes an instrument that enables the teachers to meet their own ends. True participation starts from the experiences of both the students and the teachers. A pedagogy course on mathematics will now be expected to address these issues as well.

The fascinating world of mathematics provides an unlimited scope for mathematicians to perceive problems related to three situations in the forms of concrete, abstract and intuition. However, due to abstraction and intuition some of the mathematical concepts become more complicated sometimes for teachers who are actively engaged in mathematics teaching at different stages of schooling. Thus, an exhaustive training is required to be imparted in pedagogy as well as content.

A clarification of mathematical concepts using teaching aids, experimentation, observation, practical and problem solving is required to make abstraction accessible at different stages of schooling.

Keeping in view the above concerns of school mathematics education and abstract nature of mathematics, the Department has developed this textbook on Pedagogy of Mathematics for two-year B.Ed. course. This book contains ten units covering almost all the corners of school mathematics education, including Professional Development of Mathematics Teachers. This textbook covers nature of mathematics by giving emphasis on logical and reasoning aspects alongwith aesthetic sense in mathematics thereby enriching the learners with various strategies, including lesson planning, the aims and objectives of teaching mathematics, school mathematics curriculum and learning resources in mathematics.

The content in this book was developed through a series of workshops organised by this department that involved practising B.Ed. teachers, subject experts from universities and institutes of higher learning and the members of the mathematics group of DESM. We gratefully acknowledge their efforts and thank them for their valuable contribution in our endeavour to provide good quality teaching material to student-teachers and teacher educators.

I express my gratitude to Director and Joint Director, Professor G. Ravindra, NCERT, for valuable motivation and guidance from time to time. Special thanks are also due to Dr R.P. Maurya, Associate Professor in mathematics, DESM for coordinating the programme, taking pains in editing the manuscript and making it press worthy.

We look forward for comments and suggestions for further revision and refinement of this book.

New Delhi
April 2011

HUKUM SINGH
Professor and *Head*

TEXTBOOK DEVELOPMENT COMMITTEE

MEMBERS

B.S.P. Raju, *Professor*, RIE, Mysore

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V.P. Singh, *Associate Professor*, DESM, NCERT, New Delhi

MEMBER-COORDINATOR

R.P. Maurya, *Associate Professor*, DESM, NCERT, New Delhi

ACKNOWLEDGEMENTS

The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop:

Jyoti Sharma, *Assistant Professor*, S.P.M. College, University of Delhi, Delhi; Raminder Singh, *Assistant Professor*, Department of Education, Punjabi University Regional Centre, Bhatinda, Punjab; M.L. Jaidka, *Associate Professor*, D.M.College of Education, Moga, Punjab; K. Ganeswara Roa, *Professor*, D.N.R. College of Education, Bhimvaran, Andhra Pradesh; Sharad Suare, *Lecturer*, Amrita School of Education, Bogadi, Mysore, Karnataka; Juriti Borgohain, *Lecturer*, Government Banikaunta College of Teacher Education, Haclick Nagar, Guwahati, Assam; Poonam Beniwal, *Assistant Professor*, Maharaja Suraj Mal Institute, G.G.S.I.P. University, Delhi; A.R. Aggarwal, *Associate Professor*, Regional Institute of Education, Ajmer and Vijayan K., *Assistant Professor*, D.T.E.E., NCERT, New Delhi.

Special thanks are due to Professor Hukum Singh, *Head*, DESM, NCERT for his support during the development of this book.

The Council acknowledges the efforts of Deepak Kapoor, *Incharge*, Computer Station; Shashi Devi, *Copy Editor*; Surender Kumar, Md. Iqbal Hassan, Rakesh Kumar and Ajeet Kumar; *D.T.P. Operators*; K.T. Chitralekha and Suchitra Mohanti, *Proof Reader*.

The contribution of APC-office, administration of DESM, Publication Department and Secretariat of NCERT is also duly acknowledged.

CONTENTS

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For solving a problem, you have firstly to understand the problem, secondly to devise a plan for the solution, thirdly to carry out the plan of action and fourthly to check the results.

G. Polya

UNIT
1

NATURE AND SCOPE OF MATHEMATICS

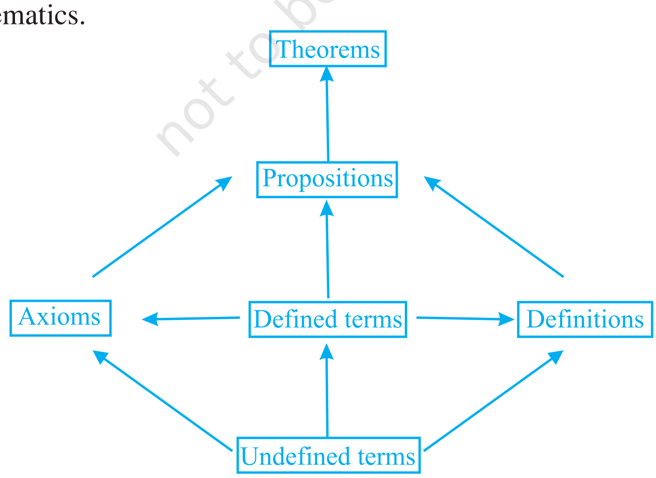
1.1 Introduction

What is Mathematics? The word 'Mathematics' has different meanings to different people. People perceive mathematics according to their own experiences and these experiences differ from person to person. Most of the people come across only certain aspects of Mathematics of which nature of mathematics has special significance. In this Unit, we give simplified version of nature of Mathematics which is nearer to the experience of most of the people to start with and then make it more and more systematic as we proceed further.

The nature of any discipline has significant impact on its pedagogy. Mathematics reveals hidden patterns that help us to understand the world around us. Thus, mathematics is a study of patterns and order dealing with numbers, geometrical objects, forms, algorithms, chance and change. More than merely being a study of arithmetic, algebra and geometry, mathematics today is a diverse discipline that deals with measurements, data analysis, observations from various fields of knowledge, inductive generalisations, proofs, logical deductions and mathematical modelling of natural phenomena, of human behaviour and of social systems. As a systematic study of abstract objects and phenomena, mathematics relies on logical reasoning rather than observation as its measure of validity of truth; yet employs observation, simulation and even experimentation as means of discovering truth. The special role of mathematics in education is a consequence of its universal applicability. In addition to theorems and theories, mathematics offers distinctive models of thought which are both versatile and powerful, including abstraction, optimisation, generalisation, logical analysis, inference and use of symbols. Mathematics empowers us to understand the information-loaded World in which we live, in a systematic and organised manner.

Most of the mathematical concepts which we have included in our school curriculum have their origin from daily life situations and happenings in the surrounding World. Majority of the basic branches of mathematics grew out of the daily life needs. For example, arithmetic and basic algebra grew out of the need for counting and other simple operations required to solve daily life problems. Human quest for possession of land and other properties needed measurement which led to the invention of geometry and trigonometry. Growth of geometry and trigonometry is also due to the curiosity and quest for understanding the Universe and happenings around us. Requirement for understanding the processes in physics led to the invention and growth of calculus. Many new branches of mathematics emerged out of necessity to solve problems faced by scientists, social scientists, commerce and trade organisations as well as warfare experts. Mathematical ideas emerging out of all these sources contribute towards the nature of mathematics.

To understand more about nature of mathematics, we should perceive relationships between its constituents. For instance, we should know the meaning of the most basic terms called 'undefined terms,' then use these undefined terms to 'define' new 'terms' and then develop 'axioms' using these terms that form the foundations of mathematical theory. This way, we arrive at collection of mathematical terms and mathematical axioms. By applying inductive reasoning, we arrive at generalisations known as propositions. We establish the truth of these propositions by using the rules of logic and call them 'Theorems'. The method of establishing the truth of a proposition and making it a theorem is known as *mathematical proof* or simply a '*proof.*' This nature of Mathematical process is known as *deductive* nature of mathematics.



Structure of Mathematics

Once the mathematical theorems are proved, one looks for applying them to solve problems in real life situations or in other branches of knowledge or in understanding a natural phenomena. Learning the skills of problem solving is very important as it involves an appropriate mathematical modelling of the problem situation, apply appropriate theorems to deduce solutions of the mathematical problem and then interpret in terms of the original problem situation. This is in brief what we are going to learn in this Unit.

Learning Objectives

After studying this Unit, the student-teachers will be able to:

- understand building blocks of mathematics viz., undefined terms, definitions, propositions, axioms, proofs, open sentences and quantifiers
- understand types of propositions, truth values of propositions and basic types of compositions of propositions
- differentiate between conjunction, disjunction, implication, equivalence and negation of propositions and write their truth tables
- understand open sentences and draw Venn diagrams for the truth sets of compound open sentences
- check logical validity of arguments
- define and identify necessary and sufficient parts of an implication
- understand the concept of a mathematical theorem and its invariants
- write converse, inverse and contrapositive of implications
- differentiate between different types of mathematical proofs
- write different types of proofs for theorems from secondary/higher secondary school mathematics
- understand that verification is not a method of proof in mathematics
- appreciate the historical development of mathematics in general and contribution of Indian mathematicians in particular
- understand aesthetic sense in mathematics and appreciate beauty in mathematics
- appreciate scope of mathematics at secondary/higher secondary school level.

1.2 Building Blocks of Mathematics

To understand the nature of mathematics, we begin to study the logical development of a mathematical system. We would first see what are various constituents or blocks of the system. In fact, undefined terms, definitions, propositions, axioms, proofs, open sentences and quantifiers are building blocks of mathematics.

1.2.1 Undefined Terms

Imagine that we are trying to learn French language and what all we have is a French to French dictionary. We look for the meaning of a French word in the dictionary. The meaning is given in terms of other French words whose meanings we do not know. Again we use the dictionary to find the meanings of these new French words and we come across more new words whose meaning we do not know. Thus, unless we know the meaning of certain minimum number of French words, there is no point looking into a French to French Dictionary to find the meaning of a new word. In the same way, unless we already know the meaning of certain minimum number of mathematical terms, we will not be able to give meaning to new mathematical terms. These members of a minimal set of mathematical terms whose meaning we take for granted without explaining in terms of other mathematical terms, and give meaning of other mathematical terms in terms of members of this minimal set of mathematical terms are known as 'undefined terms.' For example, what is a rational number? 'A rational number is of the form $\frac{p}{q}$, where p and q are integers and q is not equal to zero.' Meaning will not be known unless you know what are integers. 'So what are integers?' 'Natural numbers, zero and their negatives put together are integers.' Here unless we know what are natural numbers, zero and the negatives of natural numbers, we will not understand the meaning of integers. Do we know atleast the meaning of natural members? We only give examples of a few natural numbers, say 1, 2, 3, etc… Do these answers give complete meaning of natural numbers? Either we have to assume that we know what natural numbers are or we have to give their meaning 'axiomatically' which we will discuss later. Thus, we take a 'natural number' as an undefined term.

In geometry, point, line, plane, etc. are taken as undefined terms.

How many terms can be taken as 'undefined' in a mathematical system? Is there any rule? To answer these questions, we say that there is no definite rule. However, as far as possible, they should be minimum in number.

1.2.2 Definitions

What is a definition?. The definition of a term is a *characteristic explanation* involving terms which are either undefined terms or terms which have already been defined. The word 'explanation' here is used in the meaning of common language usage. But what is a 'characteristic explanation'? A characteristic explanation of a term is an explanation which characterises the given term, i.e., the explanation which is true for and only for the given term.

Example: 1. An equilateral triangle is a triangle in which all sides are equal.

2. Common divisor of two integers is a number which divides both the given integers.

Think! Can we define area or should we take it as an undefined term?

Many of the mathematical terms at the primary and secondary school levels, though can be defined, the definitions are beyond the scope of the students at that level. So, these terms are introduced intuitively as concepts through a process called *concept development.* Further, all the 'undefined terms' are also developed as concepts. We will study about 'concept development' at a later stage in this book.

1.2.3 Propositions

A 'Proposition' or a 'Statement' is a grammatically correct declarative sentence which makes sense, which is either true or false, but not both. If a 'Proposition' is true, we say that its truth value is 'True' and if it is false, we say that its truth value is 'False'.

For example,

'Sum of 2 and 5 is 7'

is a Proposition which is True, but

'Sum of 2 and 5 is 6'

is also a Proposition which is False. However,

'Is 7 the sum of 2 and 5?'

is not a Proposition, because it is not a declarative sentence as it does not declare anything, but is only an interrogation. 'Delhi is the capital of India' and 'River Ganga flows in England' are Propositions with truth values, 'True' and 'False', respectively.

'Find the sum of 2 and 5'

is a directive sentence and not a declarative sentence. So, it is not a Proposition. Consider,

Sum of 2 and 5 is green.

This is not a Proposition, as the sentence does not make any sense. What about the following sentence?

'Sarla is a good student.'

Is it a proposition? It is, of course, a declarative sentence which makes sense. But what about its truth value? How to verify whether Sarla is a good student or not? There are many attributes for a student to be good. You may get both 'True' and 'False' as truth values for these attributes. Also there may not be a definite answer 'True' or 'False'. The response may vary from person to person. It may also vary from situation to situation. Hence, we say that the sentence is vague and the given sentence is not a Proposition as this does not have a definite truth value.

EXERCISE 1.1

1. Which of the following terms are defined and which are undefined ?
   (a) Line (b) Points (c) Diagonal of a quadrilateral
   (d) Natural Number (e) H.C.F.
2. Which of the following are propositions?
   (a) The sum of the measures of interior angles of a triangle equals two right angles.
   (b) $2 \times 6 = 8$
   (c) Mathematics is red
   (d) An angle consists of two rays and one vertex
   (e) $2y + 3 = 5$
3. Which of the following are definitions?
   (a) A square is a quadrilateral in which all sides have equal measures.
   (b) A polynomial in x is an algebraic expression of the form $ax^3 + bx^2 + cx + d$, where, a,b,c and d are constants.
   (c) An irrational number is one which is not a rational number.
   (d) Two lines are parallel if and only if they do not intersect.
   (e) A square is a rectangle in which all the sides have equal measures.

ANSWERS

1. (c), (e) are defined and (a), (b), (d) are undefined
2. (a), (b), (d) are propositions
3. (b), (e) are definitions

1.2.4 Axioms

How do we determine whether a proposition in mathematics is true or false? We adopt a process known as logical reasoning (inductive deductive reasoning). The way of establishing the truth of a proposition is known as 'Mathematical Proof' or simply a 'Proof.' To prove a proposition, we look for already established propositions and selecting appropriate propositions among them, apply rules of reasoning to establish the truth of the given Proposition.

Now the question arises, "Can we prove or disprove every Proposition?" When we ask ourselves why is a given proposition true, we go back to our past experiences and try to give reasons in terms of other true propositions. When asked why these propositions are true, again we get back to some other true Propositions and as in case of undefined terms, we reach a stage where we are no longer able to prove or disprove a proposition, but we strongly feel that it must be true. So, we accept this proposition as true without any proof. However, we should start with a minimum number of Propositions which we assume to be true and prove any other proposition using these initial propositions (which we have assumed to be true) and rules of logic. These initial propositions whose truth we assume are known as **Axioms**. The set of Axioms of a mathematical system have the following characteristics:

1. Axioms are propositions which are assumed and accepted without any evidence to prove it.
2. Axioms are consistent.
3. Axioms are not superfluous.
4. Axioms are adequate.

The axioms are acceptable in the sense that there is no evidence to the contrary; on the other hand there are enough evidences to show that the axioms are true. They are *consistent* in the sense that by using other axioms and applying rules of logic, we should not be able to arrive at anything contrary to any of the axioms. They are not *superfluous* in the sense that we should not be able to prove any axiom using other axioms. They are adequate in the sense that it should be possible to prove any known result of the Mathematical theory with the help of the set of Axioms.

1.2.5 Proofs

Truth of a mathematical proposition is established by using the set of axioms, already established propositions and by applying the rules of logic. The method of establishing the truth of a proposition is known as *proof*. To understand the proofs and construct proofs for the propositions, we will have to first learn rules of logic or mathematical logic. After learning the rules of logic, we will again discuss more about proofs.

1.2.6 Open Sentences

Consider the sentence $x^2 + 3x + 2 = 0$. Is this sentence true or false? Unless we assign some values for the unknown or the variable, we will not be able to say whether the sentence is True or False. For example, if we assign the value 1 to x, then the sentence is false; but if we assign the value -1 to x, then the sentence is true. Thus, the sentence $x^2 + 3x + 2 = 0$ becomes a proposition on assigning real values to the unknown x. The set of all values which the unknown in the sentence is allowed to take is known as the *Universal set.* Such sentences are called open sentences (statements).

An *open sentence* is a sentence involving a variable and which becomes a proposition on substitution of values for the variable from the universal set. It is generally denoted by p_x, q_x etc.

$x + 3 < 5$, $x^2 + 3x = (2x + 1)(x + 1)$, $z^3 - 3z^2 = 5$, are some examples of open sentences.

The values of the variable from the universal set for which the open sentence becomes a true proposition is called the *Truth Set* of the open sentence. The open interval $(-\infty, 2)$ is the truth set of $x + 3 < 5$, if we take the set of real numbers as the universal set.

1.2.7 Quantifiers

Consider the following statement:

$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$, for all real numbers x.

We can rewrite this statement as

(1) For every real number x, $((x + 1)^3 = x^3 + 3x^2 + 3x + 1)$

Similarly for the statement, 'between 5 and 8, there is an integer', we can rewrite this as

(2) (There is an integer) (The integer is between 5 and 8)

The English language words 'for all' and 'there is a' used in the sentences (1) and (2) above are known as logical *quantifiers*. In mathematics, 'for all', 'for every', 'for each' have the same meaning and 'there is *a*', 'there is some', 'there exists *a*', 'for some', 'for *a*', 'there exists some *a*' have the same meaning. Symbolically, the logical quantifier 'for all' is denoted by $\forall$ and the quantifier 'there is *a*' is denoted by $\exists$. Using these symbols, we write (1) and (2) mentioned above respectively as

$(\forall x, x \text{ real})\ ((x + 1)^3 = x^3 + 3x^2 + 3x + 1)$

and

$(\exists\, x, x \text{ integer})\ (5 < x < 8)$.

Let us consider some more examples:

1. All the students in this class hail from Uttar Pradesh.
2. Sum of interior angles of every triangle is equal to 180°.
3. All prime numbers greater than 2 are odd.
4. For some integer x, $x^2 = 4$.

5. There are some students in this class hailing from Uttar Pradesh.
6. There exists a rhombus which is not a square.

Using the symbols ∀ and ∃ for the quantifiers in the above statements, we can rewrite them respectively as follows:

1. (∀ student in this class) (the student hails from Uttar Pradesh)
2. (∀ triangle) (the sum of interior angles of the triangle = 180°)
3. (∀ prime number greater than 2) (the prime number is odd)
4. (∃ integer x) ($x^2 = 4$)
5. (∃ student in this class) (the student hails from Uttar Pradesh)
6. (∃ rhombus) (the rhombus is not a square)

Many a times, statements that include quantifiers do not seem to include quantifiers. For example, consider the statement 'Sum of interior angles of a triangle is equal to 180°' or 'Diagonals of a parallelogram bisect each other'. The above statements actually stand for 'the sum of interior angles of all triangles is 180°' and 'Diagonals of all parallelograms bisect each other'. In fact, the logical quantifiers are present there in hidden form.

EXERCISE 1.2

1. Find the truth sets of (i) $x^2 + 3x + 2 = 0$, (ii) $x^2 > 0$ and (iii) $y^2 + y \leq 2y + 1$, by taking the set of real numbers as the universal set.
2. Which of the following sentences are propositions and which are open sentences? In case of open sentences, write the truth sets:
   (a) Lucknow is capital of India.
   (b) River Ganga joins Bay of Bengal.
   (c) Chalk is white.
   (d) 5 is an irrational number.
   (e) $x^2 + 3x + 2 > 5$.
   (f) $(a + 2)^2 = a^2 + 4a + 4$.
3. Write the following statements using logical quantifiers:
   (a) Every positive real number is square of a real number.
   (b) All even numbers greater than 2 are composite.
   (c) Some rectangles are squares.
   (d) There is a rational number whose square is 3.

ANSWERS

1. (i) {–1, –2}, (ii) $\mathbb{R}$ – {0}, (iii) $\left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$.

2. (a), (b), (c), (d), and (f) are propositions, (e) is open sentence.

3. (a) $\forall x$, x is positive real, (b) $\forall x$, x is even number (composite) greater than 2
(c) $\exists\ x$, (x rectangle) (x is a square), (d) ($\exists\ x$, x rational number ($x^2 = 3$)).

1.3 Calculus of Propositions

1.3.1 Simple and Compound Propositions and Sentential Connectives

A proposition which cannot be further broken into more than one proposition is known as a 'Simple Proposition' or a 'Prime Statement.'

A proposition which is not a simple proposition is known as a 'compound proposition' or a 'composite statement.'

For example, '5 is a composite number' is a simple proposition whereas '5 is a composite number or 5 divides 15' is a compound proposition using the word 'or'. 'All angles and all sides of an equilateral triangle have equal measures' is a compound proposition got by combining the two propositions 'All angles of an equilateral triangle have equal measures' and 'All sides of an equilateral triangle have equal measures' using the word 'and'. Similarly, 'after the mathematics class, I will go to the library or I will go home' is a compound proposition obtained by combining two propositions 'After the mathematics class I will go to the library' and 'After the mathematics class I will go home' by using the word 'or'.

Look at the following propositions:

1. If 15 is a composite number, then 5 divides 15.
2. I will stay at home if it rains in the evening.
3. a^2 is negative only if a is not real.
4. The square of a number is 4 if and only if the number is 2.

All the above propositions are compound propositions where in the compound propositions are formed by using words 'if … then …' '… only if …' and ' … if and only if …'

The words 'and,' 'or,' 'If … then,' 'only if …,' 'if and only if …' used to combine two propositions to form a compound proposition are called 'sentential connectives' or simply connectives. In proposition 2 above, 'if' has appeared for 'if then' and this can be easily seen by rewriting it as

'If it rains in the evening, then I will stay at home'.

1.3.2 Truth Values of a Proposition

A proposition is either 'True' or 'False.' So, the truth of proposition has two values 'True' or 'False.' These are called the truth values of a proposition.

Truth value of the proposition '5 is a composite number' is 'False' whereas the truth value of 'Every rhombus is a parallelogram' is 'True'. We will discuss more about truth values at a later stage.

1.3.3 Basic Types of Compositions of Propositions

As already discussed, two or more propositions are combined to form a compound proposition using sentential connectives. We will now discuss different basic types of composition of propositions namely conjunction, disjunction, implication or conditional, equivalence or biconditional and negation.

1.3.3.1 Conjunction

The simplest form of combining two propositions is by putting the connective 'and' between the two propositions. This is known as *Conjunction* of the two propositions.

For example, the two propositions

(i) 30 is an even number

(ii) 10 does not divide 30

can be combined using 'and' to form the proposition

(iii) 30 is an even number *and* 10 does not divide 30.

We know that proposition (i) is true whereas proposition (ii) is false. What about proposition (iii)? Before answering the question, consider the statement 'I will get up at 5 AM and go for a morning walk'. This statement is a compound statement got by two statements 'I will get up at 5 AM', 'I will go for a morning walk' using the connective 'and'. Since I have told that I will be doing both the acts, my original statement will be true only when statement 'I will get up at 5 AM' is true as well as the statement 'I will go for a morning walk' is true. In all the other cases, my original statement will be false. Similarly, proposition (iii) will be true when proposition (i) is true as well as proposition (ii) is true. In all other cases, proposition (iii) will be false. Thus, it is clear that a conjunction is true only when both component propositions are true and in all other cases the conjunction proposition will be false. Therefore, proposition (iii) is false.

If we denote the two component propositions by p and q, the compound proposition obtained by conjunction of p and q is denoted by $p \wedge q$ read p conjunction q or p cap q. From the above discussion, it is clear that in order that $p \wedge q$ is true, both p and q must be true. In all other cases $p \wedge q$ will be false. When we assign truth values T and F to p and q, the truth values of $p \wedge q$ is exhibited in the form of the following table known as truth table for $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that the truth table for $p \wedge q$ is the same as the truth table for $q \wedge p$.

1.3.3.2 Disjunction

Combining two propositions by putting connective '*or*' between the two propositions is known as *disjunction* of the two propositions.

For example, consider the proposition

(iv) 30 is an even number or 10 does not divide 30. This proposition is obtained by combining the propositions (i) and (ii) in 1.3.3.1 using the connective 'or' in between them. Is the proposition (iv) true? Before answering this, we will consider the following statement:

'I will get up at 5 AM or I will go for a morning walk' which is disjunction of the statements 'I will get up at 5 AM,' 'I will go for a morning walk.' Since in my original statement I have claimed that I will do atleast one of the two acts, my original statement will be false only when both 'I will go for morning walk' is false and 'I will get up at 5 AM' is false. In all other cases, my original statement is true. On similar lines, since '30 is an even number' is true, Proposition (iv) will be true though '10 does not divide 30' is false.

Clearly, the disjunction of two propositions will be false only when both the propositions are false. In all other cases, it will be true.

If p and q are two propositions, then the proposition obtained by disjunction of these two propositions is denoted by $p \vee q$ and the truth table for it is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that the truth table for $q \vee p$ will be the same as that of $p \vee q$.

1.3.3.3 Implication

Consider the proposition:

(v) *If* 30 is an even number *then* 10 does not divide 30.

This proposition is again obtained by composition of proposition (i) and proposition (ii) using the sentential connective 'If … then …'. Combining two propositions using "If … then …" is known as an *Implication* or *Conditional*. We say that the first proposition implies the second proposition.

Is the proposition (v) true? Before answering this question, consider the following statement:

If I get up at 5 AM, then I will go for a morning walk.

When is this statement false? Only if I get up at 5 AM and do not go for a morning walk. If I do not get up at 5 AM, whether I go for morning walk or not, my statement is not false and hence true. Thus, except for the case when I get up at 5 AM and I do not go for morning walk, my original statement is true. On similar lines, proposition (v) is false, because proposition (i) is true, but proposition (ii) is false.

An implication has in it two propositions one following 'if' and the other following 'then'. The proposition that follows 'if' is known as hypothesis and the proposition that follows 'then' is known as conclusion. An implication is false only when hypothesis is true, but the conclusion is false. In all other cases, the implication is true. Let the two propositions taken to form the implication be denoted by p and q. The *implication* 'if p then q' is denoted by $p \to q$ read p implies q. Next, $p \to q$ is false only when p is true and q is false and in all other cases $p \to q$ is true. So, the truth table of $p \to q$ is given by

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

'A number is even, if the number is divisible by 2' has the same meaning as 'If the number is divisible by 2, then the number is even', i.e., 'If p then q' has the same meaning as 'q, if p'. We denote 'q, if p' by $q \leftarrow p$ and read it as q is implied by p. Thus, $q \leftarrow p$ is the same as $p \to q$. Similarly, you can prepare the truth table for $q \to p$ and see that $p \to q$ is not always the same as $q \to p$.

Consider the statement:

'I will go for higher studies *only if* I get a distinction in the examination'.

Let us analyse this statement. Two component propositions are 'I will go for higher studies,' 'I will get a distinction in the examination' and the sentential connective is 'only if.'

Assume that 'I will get a distinction in the examination' is true and 'I will go for higher studies' is also true. Then definitely, my original statement is true. However, suppose 'I will get a distinction in the examination' is false, but 'I will go for higher studies' is true, then my original statement is false. Suppose 'I will get a distinction in the examination' is true and 'I will go for higher studies' is false, still my original statement is not false, because I had not claimed that if I get a distinction in the examination then I will go for higher studies. Clearly, if 'I will get a distinction in the examination' is false and, 'only if I will go for higher studies' is also false, then my original statement is still true. Thus, p only if q has the following truth table:

p	q	p only if q
T	T	T
T	F	F
F	T	T
F	F	T

Compare this with the truth table of $p \rightarrow q$. Both the tables are one and the same. So p only if q is logically equivalent to $p \rightarrow q$.

1.3.3.4. Equivalence or Biconditional

Consider the compound proposition:

(vi) 30 is an even number, if and only if 10 divides 30.

In fact, this is the conjunction of two propositions,

'30 is an even number if 10 divides 30,'

'30 is an even number only if 10 divides 30' with sentential connective 'and.' But we have seen in 1.3.3.3 that the above two statements can be rewritten as

(vii) 'If 10 divides 30 then 30 is an even number'

(viii) 'If 30 is an even number then 10 divides 30'

Hence, proposition (vi) is the conjunction of the two implication propositions (vii) and (viii). If we denote the two component propositions by p and q then the compound proposition is the conjunction of $p \rightarrow q$ and $q \rightarrow p$. Symbolically, we denote the compound proposition involving the sentential connective if and only if by $p \leftrightarrow q$. Thus, $p \leftrightarrow q$ is the same as $(p \rightarrow q) \wedge (q \rightarrow p)$. Notice that in order that p if and only if q is true either both p and q should be true or both p and q should be false. Otherwise $p \leftrightarrow q$ will be false. Hence, the truth table of $p \leftrightarrow q$ is as follows and it is the same as for $(p \rightarrow q) \wedge (q \rightarrow p)$ as illustrated in the truth table below:

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

1.3.3.5 Negation

Negation is not a sentential connective used for combining two propositions. But, it transforms a proposition into another and it is a powerful tool used in logical deduction to establish the truth of a statement. Consider the statements:

(ix) I will go for a morning walk

(x) I will not go for a morning walk.

Note that if the statement (ix) is true then the statement (x) is false and if the statement (ix) is false then the statement (x) is true. So, we call statement (x) as the negation of sentence (ix). For a statement p, its negation is denoted by $\sim p$ and read negation p. p is true whenever $\sim p$ is false, and p is false whenever $\sim p$ is true. Further, it is clear that $\sim(\sim p)$ is p.

The truth table for $\sim p$ is given by

p	$\sim p$
T	F
F	T

Consider the compound statement,

'Either I will play chess or I will study.'

This statement is false only when I will not play chess and I will not study also. So, the negation of $p \vee q$ is $(\sim p) \wedge (\sim q)$.

Similarly, consider the statement,

'I will play chess and I will study.'

This statement will be false if either I will not play chess or I will not study. So the negative of $p \wedge q$ is $(\sim p) \vee (\sim q)$. These facts are clear from the following truth tables:

p	q	$\sim p$	$\sim q$	$p\vee q$	$\sim(p\vee q)$	$(\sim p)\wedge(\sim q)$	$p\wedge q$	$\sim(p\wedge q)$	$(\sim p)\vee(\sim q)$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

From the above tables

$\sim(p\vee q) \leftrightarrow (\sim p)\wedge(\sim q)$

and

$\sim(p\wedge q) \leftrightarrow (\sim p)\vee(\sim q)$

These statements are known as De Morgan's Laws.

Think! Is $\sim(p\rightarrow q)$ equivalent to $(\sim p)\wedge q$ or to $p \wedge (\sim q)$?

1.4. Calculus of Open Sentences and Venn Diagrams

1.4.1 Conjunction of Two Open Sentences

Let p_x and q_x be two open sentences and let U be the universal set in which the variable x takes the values. Then the conjunction of p_x and q_x denoted by $p_x \wedge q_x$ is the open sentence which at an element $a \in$ U is the conjunction $p_a \wedge q_a$ of the propositions p_a and q_a. Let P and Q be the truth sets of p_x and q_x respectively. Then an element $a \in$ U is in the truth set of $p_x \wedge q_x$ iff $p_a \wedge q_a$ is true, i.e., if and only if (iff) a is in the truth sets of both p_x and q_x, i.e., iff $a \in$ P and $a \in$ Q, i.e., iff $a \in$ P $\cap$ Q. Hence truth set of $p_x \wedge q_x$ is P $\cap$ Q, the intersection of truth set of p_x and truth set of q_x. In terms of the Venn diagram, we can represent it as shown in Fig.1.1.

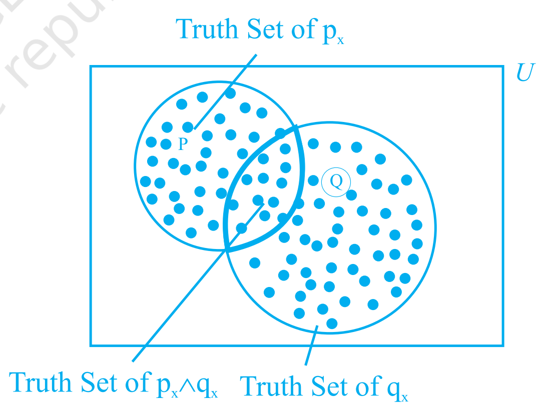


Fig. 1.1

1.4.2 Disjunction of Two Open Sentences

Let p_x and q_x be open sentences with universal set U. Then the disjunction $p_x \vee q_x$ of the open sentences p_x and q_x is the open sentence which at an element $a \in$ U is the proposition

$p_a \vee q_a$, the disjunction of the propositions p_a and q_a. Let P and Q be the truth sets of the open sentences p_x and q_x, respectively. Then an element $a \in$ U is in the truth set of $p_x \vee q_x$ *iff* $p_a \vee q_a$ is true, i.e., iff either p_a is true or q_a is true, i.e., iff either a is in the truth set of p_x or a is in the truth set of q_x, i.e., iff either $a \in$ P or $a \in$ Q, i.e., iff $a \in$ P$\cup$Q. So, the truth set of $p_x \vee q_x$ in P$\cup$Q, the union of truth set of p_x and truth set of q_x. Thus, the Venn diagram of disjunction of two open sentences is given in the Fig 1.2.

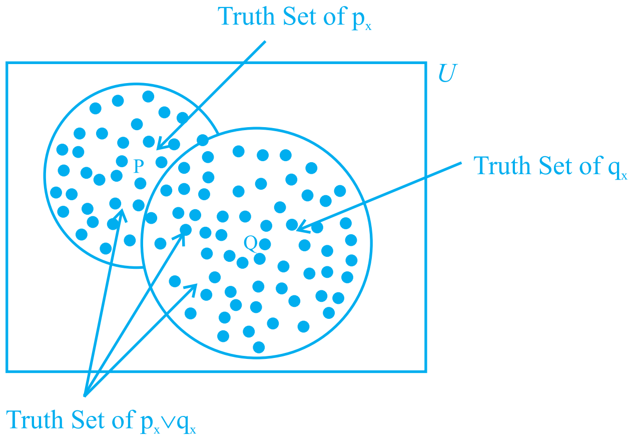


Fig. 1.2

1.4.3 Implication of Open Sentence

Let p_x and q_x be open sentences with universal set U. Then the implication $p_x \rightarrow q_x$ is the open sentence which is the implication proposition $p_a \rightarrow q_a$, where p_a and q_a are the propositions corresponding to open sentences p_x and q_x at $a \in$ U. So, an element $a \in$ U is in the truth set of $p_x \rightarrow q_x$ *iff* a is such that $p_a \rightarrow q_a$ is true, i.e., iff a is such that $p_a \rightarrow q_a$ is not false, i.e., iff a is such that $\sim(p_a \rightarrow q_a)$ is false, i.e., iff a is such that $p_a \wedge (\sim q_a)$ is false (Note that $\sim(p \rightarrow q)$ is logically equivalent to $p \wedge (\sim q)$, i.e., iff a is not in the truth set of $p_x \wedge (\sim q_x)$, i.e., iff $a \notin$ P$\cap$(Q^c), i.e., iff $a \in$ (P$\cap$Q^c)^c, i.e., iff $a \in$ P$^c\cup$Q.

Thus, the Venn diagram of implication of open sentence is given by Fig 1.3.

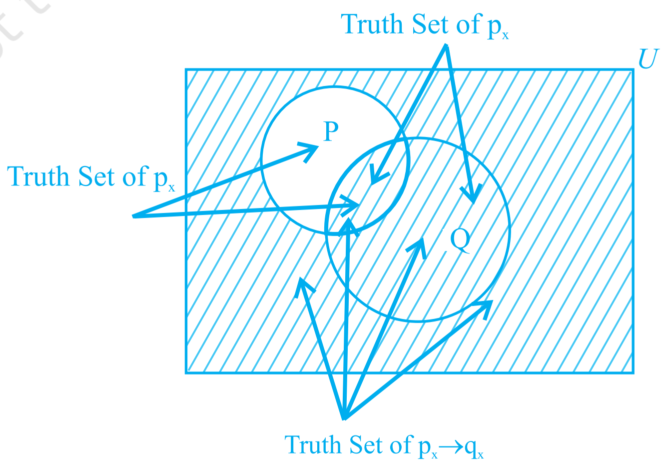


Fig. 1.3

1.4.4 Equivalence of Open Sentence

Consider two open sentences p_x and q_x with universal set U. Then the open sentence p_x if and only if q_x is the open sentence which at $a \in$ U is the proposition $p_a \leftrightarrow q_a$. We denote it by $p_x \leftrightarrow q_x$. What is the truth set of

$p_x \leftrightarrow q_x$ if P and Q are truth sets of p_x and q_x? An element $a \in U$ is in the truth set of $p_x \leftrightarrow q_x$ iff $p_a \leftrightarrow q_a$ is true, i.e., iff $p_a \rightarrow q_a$ is true and $q_a \rightarrow p_a$ is true (Note that $p \rightarrow q$ is logically equivalent to $(\sim p) \vee q$ and $q \rightarrow p$ is logically equivalent to $p \vee (\sim q)$), i.e., iff $a \in P^c \cup Q$ and $a \in Q^c \cup P$, i.e., iff $a \in (P^c \cup Q) \cap (P \cup Q^c) = (P \cap Q) \cup (P^c \cap Q^c)$. Hence, the truth set of $p_x \leftrightarrow q_x$ is $(P \cap Q) \cup (P^c \cap Q^c)$ [see Fig. 1.4].

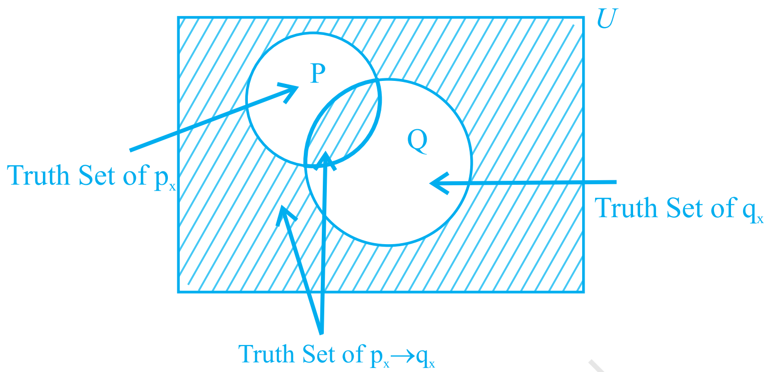


Fig. 1.4

1.4.5 Negation of an Open Sentence

Consider an open sentence p_x with universal set U. Then the negation of p_x is the open sentence which at $a \in U$ is the proposition $\sim p_a$. Negation of p_x is denoted by $\sim p_x$.

Hence, an element $a \in U$ will be in the truth set of $\sim p_x$ iff $\sim p_a$ is true, i.e., iff p_a is false, i.e., iff a is not in the truth set of p_x, i.e., iff $a \notin P$, i.e., iff $a \in P^c$. Thus, the truth set of $\sim p_x$ is P^c [see Fig 1.5].

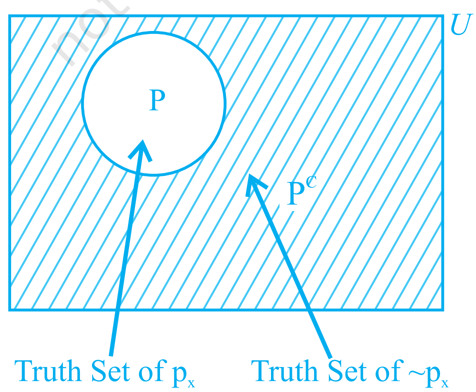


Fig. 1.5

1.5 Logical Validity of Conclusions

An argument is an assertion of the truth of a statement subject to assuming the truth of a set of statements, $\{s_1, s_2, ..., s_n\}$. For example,

(1) If I get up at 5 AM in the morning, I will go for a morning walk.

(2) If I do not go for morning walk, I will not be able to take my breakfast.

(3) I took my breakfast.

Therefore,

(4) I have got up at 5 AM in the morning is an argument.

In the above argument, the first three statements which are coming before 'therefore,' are called premises and the last sentence following the word 'therefore' is called the conclusion. The conclusion is said to be a logically valid consequence of the premises, if whenever all the premises are true, the conclusion is also true. In this case, we say that the argument is *logically valid*.

We will now see whether the above argument is logically valid or not. To see this we should find out whether (4) is true when (1) to (3) are all true. Now, since 'I took my breakfast' is true, I will not be able to take my breakfast is false. Therefore, 'I will not go for morning walk' is false. Hence, 'I will go for morning walk' is true. So, (1) is true whether I get up at 5 AM in the morning or not. Therefore, there is no guarantee whether (4) is true or false. Therefore, the conclusion of the above argument is not logically valid.

Logical validity of the conclusion of an argument can also be checked by writing the truth table as explained below:

Let us denote by:

p : I get up at 5 AM in the morning,

q : I go for morning walk,

r : I take my breakfast.

Here,

$s_1 : p \rightarrow q$

$s_2 : (\sim q) \rightarrow (\sim r)$

are premises and

$s : p$

is the conclusion.

The truth table of premises and conclusion is given below :

			Premises s_1	s_2	Conclusion s	
p	q	r	$p \to q$	$(\sim q) \to (\sim r)$	p	
T	T	T	T	T	T	critical row
T	T	F	T	T	T	critical row
T	F	T	F	F	T	
T	F	F	F	T	T	
F	T	T	T	T	F	critical row
F	T	F	T	T	F	critical row
F	F	T	T	F	F	
F	F	F	T	T	F	critical row

The conclusion of the argument is logically valid if all the premises are true, then conclusion takes the truth value true. The rows of the truth table, where all the premises are true, are known as critical rows. Note that in the last three critical rows, truth value of conclusion is false. Hence, the given argument is logically invalid.

However, if we change the first premise above by

1. I will go for morning walk only if I get up at 5 AM in the morning then (4) in fact is a logically valid conclusion of (1), (2) and (3). Because, since (3) is true, I will not be able to take my breakfast is false. As (2) is true, it follows that I do not go for morning walk is false. Therefore, I go for morning walk is true. Hence, by truth of (1) it follows that I have got up at 5 AM in the morning is true. Hence, (4) is true.

EXERCISE 1.3

Write the premises and conclusion of the following arguments and then test their validity using truth tables or otherwise:

1. If I shift my house, I will not be able to go on Holiday in June.

   I am going on Holiday in June.

   Therefore,

   I will not shift my house.

2. If I work hard and I do well in the interview, then I will get my promotion.

I have not got my promotion.

So,

I have not done well in the interview.

(What happens if we replace 'and' by 'or' in the first statement?)

3. If the prize giving ceremony runs late then the cultural programme will delay.

If the cultural programme delays, I will miss my dinner.

I did not miss my dinner.

Hence,

The prize giving ceremony did not run late.

ANSWERS

1. Valid 2. Not Valid 3. Valid

1.6 Necessary and Sufficient Conditions in an Implication

Consider the implication. 'If a natural number p greater than 2 is a prime, then p is an odd natural number.' If the above implication is true, (in fact it is true) then 'p is a prime number greater than 2' guarantees the truth of 'p is an odd natural number.' If p is not a prime number greater than 2, then it is even and hence, a multiple of 2 and so, we can immediately conclude that p is not a prime greater than 2. So, for a natural number greater than 2 to be a prime, it is necessary that it must be an odd natural number. Thus, we say that 'p is an odd natural number' is a necessary condition for 'a natural number greater than 2 to be a prime number'. But, 'p is an odd natural number' does not guarantee 'p to be a prime number.' Hence, that 'p is an odd natural number' is not a sufficient condition for 'p to be a prime number greater than 2'. However, if 'a natural number p greater than 2 is a prime number' is true, then definitely 'p is an odd natural number' is also true. Thus, the truth of 'a natural number p greater than 2 is a prime number' guarantees or is sufficient to conclude the truth of 'p is an odd natural number.' Hence, 'a natural number p greater than 2 is a prime number' is a sufficient condition for 'p to be an odd natural number.'

In general, in an implication $p \to q$, q is necessary condition for p and p is a sufficient condition for q.

In case of a biconditional $p \leftrightarrow q$, it is equivalent to $p \to q$ and $q \to p$. We generally refer $p \to q$ as the necessary part of $p \leftrightarrow q$ and $q \to p$ as the sufficient part. In fact, when we refer $p \to q$ is the necessary part, we mean that q is a necessary condition for p to be true and when we refer $q \to p$ as the sufficient part, we mean that q is a sufficient condition for p to be true.

For example, consider the statement

'A triangle ABC is right angled at A if and only if $BC^2 = AB^2 + AC^2$. Here, assuming the truth of 'triangle ABC to be right angled at A' and establishing the truth of '$BC^2 = AB^2 + AC^2$' we call as the necessary part and then establishing the truth of 'triangle ABC is right angled at A' by assuming the truth of '$BC^2 = AB^2 + AC^2$,' we call it as the sufficient part.

Instead of stating that "A triangle ABC is right angled at A if and only if $BC^2 = AB^2 + AC^2$" we say that "A necessary and sufficient condition in order that a triangle ABC is right angled at A is that $BC^2 = AB^2 + AC^2$."

Think! In the statement 'If a triangle is equilateral then it is equiangular.' Clearly 'The triangle is equiangular' is a necessary condition for 'the triangle to be equilateral.' Is it a sufficient condition also?

1.7 A Mathematical Theorem and its Invariants

1.7.1 A Mathematical theorem is a logically valid conclusion drawn from a set of premises, axioms and already established theorems of Mathematical system.

Theorems are usually in the form of an implication (or a biconditional) with premises followed by connective 'if then' (or if and only if). For example,

1. 'If *p* is a prime number and *p* divides *ab* then either *p* divides *a* or *p* divides *b*',

   is an example of a theorem in which '*p* is a prime and *p* divides *ab*' are premises and the conclusion is 'either *p* divides *a* or *p* divides *b*'.

2. 'If in a quadrilateral, the diagonals bisect each other then the quadrilateral is a parallelogram' is an example of a theorem in which 'In a quadrilateral the diagonals bisect each other' is the premise and the conclusion is 'the quadrilateral is a parallelogram'.

   However, the statement of certain theorems may not look like implications. For example, consider the statement:

3. 'In a parallelogram the diagonals bisect each other' which we consider as a theorem. In fact here, what is given is only the conclusion. The premise is not visible. However, we can restate the above statement in the form of an implication as follows:

3′. 'If a quadrilateral is a parallelogram, then its diagonals bisect each other' in which the premise is 'the quadrilateral is a parallelogram' and conclusion is 'the diagonals of the quadrilateral bisect each other'.

   Or look at the statement.

4. 'Sum of the measures of interior angles of a triangle is 180°' which is also a theorem. This can be restated as follows:

4′. If $\hat{A}$, $\hat{B}$ and $\hat{C}$ are the angles of a triangle, then their sum is 180°.

Similarly 'Every integer greater than one is a product of powers of primes' is a theorem which can be restated as' If n is an integer greater than 1 then it is a product of powers of primes.'

As observed earlier, a theorem can be a biconditional also. For example consider,

5. 'A quadrilateral is a parallelogram if and only if its diagonals bisect each other'.

In fact, this is conjunction of the two implications

5′. 'If a quadrilateral is a parallelogram, then its diagonals bisect each other' and

5″. 'If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram'.

The method of establishing the validity of a theorem is known as Mathematical Proof or simply a proof. By assuming the truth of the premises, truth of the axioms of the mathematical system and the truth of already proved theorems, we show that the conclusion of the theorem is logically valid.

1.7.2 Converse, Inverse and Contrapositive of an Implication

Consider the theorem,

1. If n is a prime number greater than 2, then n is an odd natural number.

The statement got by interchanging the premise and the conclusion.

1′. If n is an odd natural number then n is a prime number greater than 2.

The statement obtained by interchanging the premise and conclusion of a theorem is called the converse of the theorem. Thus, 1′ is the converse of theorem 1 above. In general, for any theorem $p \rightarrow q$, where p is the premise(s) and q is the conclusion, the proposition $q \rightarrow p$ is called its converse. As 1′ is not true, where as 1 is true, it follows that the converse of a theorem, need not be valid. This is because $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent. However, the converse of the theorem,

2. 'If a quadrilateral is a parallelogram, then its diagonals bisect each other' is a true proposition

2′. 'If the diagonals of a quadrilateral bisect each other, then it is a parallelogram' and is in fact true and hence, is a theorem.

The statement,

1″. 'If n is not a prime greater than 2, then n is not an odd natural number' is got by negating both the premise and conclusion of Theorem 1.

A proposition obtained by negating the premise and conclusion of a theorem is called the inverse of the theorem. If $p \rightarrow q$ is the theorem, then its inverse proposition is $(\sim p) \rightarrow (\sim q)$.

You see that 1″ is not true. But the inverse of Theorem 2,

2″ If a quadrilateral is not a parallelogram, then its diagonals will not bisect each other is, in fact, true and hence, is a theorem.

In general, inverse of a theorem need not be true. This is because the propositions $p \rightarrow q$ and $(\sim p) \rightarrow (\sim q)$ are not equivalent.

Consider the statement:

1‴ If n is not an odd natural number, then n is not a prime number greater than 2 by replacing the premise by the negation of the conclusion and the conclusion by the negation of the premise.

The proposition got by replacing in a theorem, its premise by the negation of its conclusion and the conclusion by the negation of its premise is called the contrapositive of the theorem.

If $p \rightarrow q$ is the theorem, the proposition $(\sim q) \rightarrow (\sim p)$ is its contrapositive.

By the truth tables of $p \rightarrow q$ and $(\sim q) \rightarrow (\sim p)$,

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(\sim q) \rightarrow (\sim p)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

it is seen that $p \rightarrow q$ and $(\sim q) \rightarrow (\sim p)$ are logically equivalent, i.e., whenever $p \rightarrow q$ is true, $(\sim q) \rightarrow (\sim p)$ is true and whenever $(\sim q) \rightarrow (\sim p)$ is true, $p \rightarrow q$ is true.

Hence, whenever a theorem is true, its contrapositive proposition is also true and when its contrapositive is true, the theorem is true. So, to prove a theorem it is sufficient to prove its contrapositive.

EXERCISE 1.4

Write the converse, inverse and contrapositive of the following theorems:

1. In a triangle, if two sides are equal, then the angles opposite to equal sides have equal measures.
2. If the three sides of one triangle are equal in length respectively to the three sides of another triangle, then the three angles of one triangle are equal in measure respectively to the three angles of the other triangle.

ANSWERS

1. (i) Converse: In a triangle, if the angles opposite sides have equal measures, then the two sides are equal.

(ii) Inverse: In a triangle, if two sides are not equal, then the angles opposite to these two sides do not have equal measures.

(iii) Contrapositive: In a triangle, if its measures of angles opposite to two sides are not equal, then the two sides are not equal.

2. (i) Converse: If the three angles of one triangle are equal in measure respectively to the three angles of another triangle, then the three sides of the first triangle are respectively equal to the three sides of the other triangle.

(ii) Inverse: If it is not true that the three sides of one triangle are equal in length to the three sides of another triangle, then it is not true that the three angles of one triangle are not equal in measure respectively to that of the other triangle.

(iii) Contrapositive: If it is not true that the three angles of one triangle are equal in measure respectively to the three angles of another triangle, then it is not true that the three sides of one triangle are equal in length to the three sides of the other triangle.

1.7.3. Proof of Mathematical Theorems

Method of establishing the logical validity of the conclusion of a theorem as a consequence of the premise, axioms, definitions and already established theorems of the mathematical system is known as 'proof of the theorem' or simply 'a proof.'

For example, consider the theorem:

If x and y are rational numbers, then $x + y$ is a rational number. Here, what is the premise? It is that x and y are rational numbers. What is the conclusion to be established? It is that $x + y$ is a rational number.

To prove the theorem, we start with the premise, x and y are rational numbers.

Hence, there exist integers p, q with $q \neq 0$ such that $x = \frac{p}{q}$

and there exist integers r, s with $s \neq 0$ such that $y = \frac{r}{s}$.

Hence, $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$ (by definition of addition of rational numbers)

Since, p, q, r, s are integers, ps, qr and qs are integers (already established theorems for integers)

So, $ps + qr$ is an integer (already established theorem for integers).

Also $q \neq 0$, $s \neq 0$, hence, $qs \neq 0$ (already established theorem for integers).

Therefore, $x + y = \frac{ps + qr}{qs}$ is a rational number.

The above argument immediately following the statement of the theorem is the proof of the theorem.

Think! How to prove that the product of two rational numbers is a rational number?

1.7.4 Types of Proofs

There are several types of proofs in Mathematics.

1.7.4.1 Direct Proof

A direct proof of a theorem after writing it in the form of an implication, is the most commonly used way of proof in which we start with the premise as true and applying rules of logic, axioms, definitions and already established theorems in which premise holds good, show that the conclusion is true. For example, in 1.7.3, to establish the validity of the theorem 'If x and y are rational numbers, then $x + y$ is a rational number' the proof given is a direct proof. We give one more example for direct proof. Consider the theorem,

In a parallelogram, the diagonals bisect each other.

In the implication form, the statement is equivalent to:

If a quadrilateral is a parallelogram, then its diagonals bisect each other.

The premise is 'The quadrilateral is a parallelogram' and the conclusion is 'Its diagonals bisect each other.'

We start with a quadrilateral ABCD which is a parallelogram.

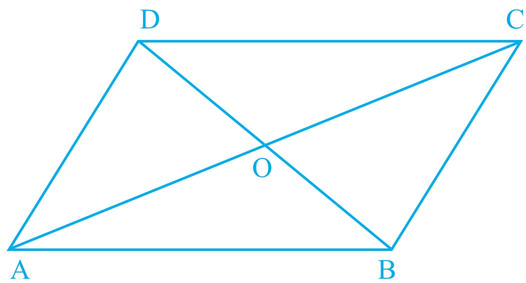


Fig. 1.6

Let the diagonals AC and BD meet at O.

Since ABCD is a parallelogram AB = CD (Opposite sides of a parallelogram are equal in length, already established theorem), AB $\parallel$ CD (by definition of a parallelogram) and AC is a transversal.

Therefore, $\angle$ ACD = $\angle$ CAB (If a transversal cuts a pair of parallel lines, then the alternate angles are equal, already established theorem)

But, AB $\parallel$ CD and BD is a transversal.

Therefore, $\angle$ CDB = $\angle$ DBA (If a transversal cuts a pair of parallel lines, then the alternate angles are equal, already established theorem)

Therefore, $\angle$ CDO = $\angle$ OBA

Therefore, in Δ ODC and Δ OBA

CD = AB

$\angle$ OCD = $\angle$ OAB

$\angle$ CDO = $\angle$ OBA

Therefore, Δ ODC $\cong$ Δ OBA (ASA congruence theorem, already established)

Therefore, OD = OB (by definition of congruence of triangles)

OC = OA

Therefore, AC and BD bisect each other.

1.7.4.2 Indirect Proofs

Proof by Contrapositive: This method of proof is based on the fact that a proposition $p \rightarrow q$ is equivalent to its contrapositive $(\sim q) \rightarrow (\sim p)$. So, to prove a theorem using contrapositive, we assume that the negation of the conclusion is true and in a straight forward way show that the negation of the hypothesis is true, i.e., if the conclusion is false, then premises are false. But, the premises are true, therefore, the conclusion must be true.

For example consider the theorem:

'If n^2 is odd then n is odd'

Its contrapositive is 'If n is not odd then n^2 is not odd'. To prove the contrapositive, assume that n is not odd. So, n is even. Therefore $n = 2m$ for some integer m (by definition of even number). Therefore $n^2 = (2m)^2 = 4m^2 = 2\ (2m^2)$ is even. Hence, n^2 is not odd. Therefore, if n^2 is odd, then n is odd.

Think! Can you think of a contrapositive proof from Secondary School Geometry?

Proof by Contradiction (or reductio ad absurdum): This type of proof is used normally, when it is not straight forward as to how to proceed with the proof. We assume to the contrary that the conclusion is false and by logical arguments arrive at something which is absurd. Hence, our assumption was wrong. Hence, the conclusion is true.

For example, consider

'There is no rational number whose square is 2.'

We prove this, by assuming to the contrary that there is no rational number whose square is 2, is false, i.e., we assume that there is a rational number $\frac{p}{q}$ whose square is 2. (By cancelling common factors in p and q, if any, we can assume that p and q have no common factors other than 1). Now $\left(\frac{p}{q}\right)^2 = 2$. Therefore, $\frac{p^2}{q^2} = 2$. So, $p^2 = 2q^2$. Hence, p^2 is even, therefore, p is even, (Prove!). Therefore $p = 2m$ for some integer m. Hence, $p^2 = 4m^2$. So, $2q^2 = 4m^2$ or $q^2 = 2m^2$. Hence, q^2 is even. Therefore, q is even. Thus, both p and q are even and hence, 2 is a common factor for p and q, a contradiction to our assumption that p and q have no common factors. So, our assumption to the contrary that there is a rational number whose square is 2, is false. Hence, there is no rational number whose square is 2.

Think! Can you show by reductio ad absurdum that 'If the square of a natural number is a multiple of 3, then the number itself is a multiple of 3'.

1.7.5. Method of Disproof or Counter Example

If we have a statement involving a universal quantifier, to prove that the statement is true, we have to show that the statement is true for every element in the universal set. However, to show that such a statement is false, it is sufficient to show that there is a particular value in the universal set for which the statement is false. Exhibiting such a value for which the statement is false is known as counter example. Method of giving such a counter example to show that such a statement is false, is known as *method of disproof.*

For example, consider the statement, 'Every odd natural number is a prime.'

To disprove this statement, it is sufficient if we can give example of one odd natural number which is not a prime. Consider the natural number 9. It is odd as $9 = 2 \times 4 + 1$. However, it is not a prime since $9 = 3 \times 3$. So, 9 is an odd natural number which is not a prime. Hence, the given statement is not true.

1.7.6. Difference between Proofs and Verifications

Consider the statement

$1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$ for every natural number n.

If we put $n = 1$, L.H.S. = 1 and R.H.S. = $\dfrac{1(1+1)}{2} = 1$. Therefore, L.H.S. = R.H.S.

If we put $n = 2$, L.H.S. = $1+ 2 = 3$ and R.H.S. = $\dfrac{2(2+1)}{2} = 3$.

Therefore, L.H.S. = R.H.S.

If we put $n = 3$, L.H.S. = $1+ 2 + 3 = 6$ and R.H.S. = $\dfrac{3(3+1)}{2} = 6$.

Therefore, L.H.S. = R.H.S.

If we put $n = 10$, L.H.S. = $1+ 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

and R.H.S. = $\dfrac{10(10+1)}{2} = 55$. Therefore, L.H.S. = R.H.S.

When we have a statement involving a universal quantifier, finding whether the statement is true or false by taking particular values for the variable in the universal set, is known as verification.

If the universal set is finite, it is possible to verify the truth of the statement for each and every value of the variable in universal set, and hence, establish the validity of the statement. However, if the universal set is infinite, though we can verify the truth of the statement for a very large number of values of the variable in the universal set, we will not be able to verify this way for all values of the variable in the universal set though verifying the truth of the statement for all values of the variable when the universal set is finite is a proof of the statement. Verification will not be proof when the universal set is infinite.

However, in the secondary school, many a time when a proposition is stated involving a universal quantifier with set of natural numbers as the universal set, we verify the truth of proposition by verifying it for $n = 1, 2, 3, 4$ and then say that 'So the proposition is true.' Verifying for a larger number of values only gives more evidence for the proposition to be true and does not establish the truth of the proposition. For example, consider the proposition. '$n^2 - n + 41$ is a prime number for every natural number n?'

By puting $n = 1, 2, 3, \ldots$ upto 40, every time we can verify that $n^2 - n + 41$ is a prime number. So, this leads us to believe strongly that $n^2 - n + 41$ is a prime number for every natural number n. However, this is not true as by putting $n = 41$ we get $n^2 - n + 41 = 41^2 - 41 + 41 = 41^2$ which is not a prime. Though for $n = 1$ to 40 the proposition is verified, one counter example viz. $n = 41$, is sufficient to disprove the given proposition.

EXERCISE 1.5

1. Prove that the square of an odd number is odd both by direct and indirect methods.
2. Prove that if two lines are intersected by a transversal, so that a pair of alternate angles are equal, then the lines are parallel by indirect method.
3. Show by method of contrapositive that for a natural number n, if $n^2 > 36$, then $n > 6$.
4. Show by method of contrapositive that if $x^2 - 1 < 0$, then $-1 < x < 1$.
5. Disprove the statement 'Every natural number is less than the sum of its proper divisors'
6. Give an example to show that verification is not a proof.

1.8 History of Mathematics

Invention and development of mathematical concepts started as early as the prehistoric age. Baboon's Fibula dating back to approximately 35000 BC*, discovered in the Lebombo Hills of Swaziland, has 29 distinct notches deliberately cut into it suggesting counting to be prevalent at that time. There were attempts to quantify time as early as 35000 BC – 20000 BC as suggested by the artifacts discovered in Africa and some parts of Europe. It is believed that the marks carried on the Ishango Bone found in Northern Congo belonging to this period demonstrate either a sequence of prime numbers or a six month lunar calendar. There are evidences that in around 5000 BC, Egyptians pictorially represented geometric figures. The monolithic designs dating back to 3000 BC found in Britain have geometric figures of circles, ellipses, etc. in them.

* BC may be read as BCE

Though some mathematical concepts were in the usage in the prehistoric period itself, mathematics in written form dates back only to around 2500 BC, the period of Ancient Summerians (or Babylonians). Clay tablets recovered from Mesopotomia have multiplication tables inscribed in them as well as certain problems and geometrical exercises. Writings on the clay tablets of later years belonging to 1800 to 1600 BC have treatment of fractions, algebra, methods for solving linear and quadratic equations etc. They used sexagesimal numerical system with base 60.

A systematic account of study of mathematics in the form of an instructional manual to the students in arithmetic and geometry came in the form of Egyptian mathematical text 'Rhind Papyrus' dating back to 1650 BC. It gives methods for multiplication, division, working with unit fractions, formula for area, prime and composite numbers, sieve of Eratosthenes for primes, perfect number, arithmetic mean, geometric mean and harmonic mean. It also gives a method to solve first order linear equations and methods to find the sum of arithmetic and geometric series.

A milestone in the development of mathematics was the development of much more sophisticated mathematics by the Greeks. All the earlier development of mathematics depended on inductive reasoning, and it was Greeks perhaps for the first time used deductive reasoning in mathematics. Greeks used definitions, axioms and rigour of mathematical deductions using logic. Thales of Miletus (624-546 BC) and Pythagoras of Samos (582 - 507 BC) were the pioneers of Greek Mathematics. Thales is said to have used deductive reasoning for the first time in geometry in deriving four corollaries to the famous theorem named after him. He used geometry to find the heights of pyramids and also distance of ships from the sea shore. Pythagoras who established Pythagorean school coined the word mathematics. He is credited with the first proof of Pythagorean theorem though the statement of this theorem along with the existence of irrational numbers was known to mathematicians of many other cultures. It was Aristotle (384 - 322 BC) who wrote down the laws of logic. Euclid (around 300 BC) gave a systematic format for the study of geometry using definitions, axioms, theorems and proofs. He created a record of all the results in mathematics known till his time in the form of his famous ' The Elements,' a volume of books. Archemedes (287-212 BC) used method of exhaustion and summation of infinite series to calculate area under the arc of a parabola. He also gave a very good approximation to *pi*(π).

Dating back to 300 BC, Chinese Mathematics is very different from mathematics from other cultures. They used a decimal position and notation system. It had distinct symbols for digits from 1 to 10 and additional symbols for higher powers of 10. For examples 2356 was written as 'symbol for 2 followed by symbol for 10^3; symbol for 3 followed by symbol for 10^2; symbol for 5 followed by symbol for 10; lastly symbol for 6.' This method of representing numbers is known as method of rod numerals which perhaps existed much before the modern Hindu numeration system. Pioneering work in Chinese Mathematics came during the 13th century in the form of Chinese algebra dealing with solution of simultaneous higher order algebraic equations and binomial expansion upto eighth power.

Indian Mathematics flourished with geometry as early as Indus Valley Civilisation dating back to 2600 - 1900 BC. But, no mathematical documents of this era are available. We will discuss the development of Indian Mathematics separately in the next section.

Contributions to mathematics were also significant during the Islamic empire spreading across Persia, Middle East, Northern Africa, Central Asian Countries and a part of India during eighth century AD*. Persian Mathematician Khwarizmi, often called the father of algebra, wrote several important books like 'On the calculations with Hindu Numerals' (written around 825 AD), which was instrumental in spreading Hindu Mathematics and Hindu Numerals to the Western World. He gave explanations for algebraic solutions of quadratic equations with positive roots. He used method of 'reduction' and 'balancing' for solving algebraic equations. It was Al-Karaji around 1000 AD who gave first known proof by Principle of Mathematical Induction in one of his books for Binomial Theorem and also for sum of cubes of first *n* natural numbers, Ibual-Haythem derived a formula for sum of fourth powers of natural numbers which is readily generalisable to sum of all positive integral powers of natural numbers. Omar Khayyam laid foundations for analytical geometry and non-euclidean geometries. Sharafal-Din al-Tusi in the 12th century introduced concept of functions and also found 'derivatives' of cubic polynomials. In 13th century Nasir al-Bin-Tusi made advances in spherical trigonometry. However, from 15th century onwards there was no significant contributions from Islamic Mathematics.

In the early middle ages, in Europe, Boethius was responsible for inclusion of Mathematics in the curriculum, covering the study of arithmetic, geometry, astronomy and music. His book 'De-Institutione Arithmaticae' is a translation of the Greek Mathematician Nicomacthu's 'Introduction to Arithmetic' and also excerpts from Euclid's 'Elements.' In the 12th century, European Scholars who travelled Spain and Sicily came across latin translations of Arabic Mathematics in general and Khwarizmi's 'The Compedium book on calculations by Completion and Balancing' in particular and also complete text of Euclid's 'Elements'. These sources introduced Hindu Numerals to Europe and gave a new direction leading to significant development of mathematics in Europe. New mathematical concepts were developed during 14th century for the solutions of various problems including that of motion.

Explosive break through in the development of mathematics came during 17th century due to the works of Galileo, Tycho Brache and John Kepler in their attempt to study motion of the planets. Kepler's laws of planetary motion gave mathematical laws for the motion of planets. Analytical geometry was developed by French Mathematician Rene' Descartes

* AD may be read as CE

there by making it possible to plot the orbits of planets on graph papers. It was Simon Steyin of this century who was responsible for the modern decimal notations both for rational and irrational numbers.

English mathematician Isaac Newton and German mathematician Gottfried Leibniz independently developed calculus and even today mathematicians use the notations of calculus as introduced by Leibniz. During 17th century, new areas of mathematics were developed. Pierre de Fermat and Blaise Pascal were responsible for the development of probability theory and combinatorics.

Eighteenth century saw the emergence of one of the most influential mathematicians of all times namely Leonhard Euler. It was he who standardised mathematical terms and notations, including the symbol i for the complex number whose square is -1 and the symbol π for the ratio of the circumference of a circle to its diameter. His main contribution, includes his works in the area of topology, graph theory, calculus, combinatorics and complex analysis. During this century another famous mathematician Joseph Louis Lagrange did pioneering works in number theory, algebra, differential calculus and calculus of variations.

It was during nineteenth century that the treatment in mathematics became more and more abstract. Carl Fredric Gauss (1777 - 1855) did revolutionary and remarkable work in the study of functions of a complex variable, geometry of surface and the study of convergence of infinite series. He gave the first satisfactory proof of Fundamental Theorem of Algebra.

During nineteenth century Lobachevsky and Bolyai independently showed that the parallel postulate in Euclidean geometry need not be assumed to be true leading to the development of two non-Euclidean geometries namely Hyperbolic Geometry and Elliptic Geometry. It was during this century that Riemann developed Riemannian Geometry which generalises all three form of geometries-Euclidean, Hyperobolic and Elliptic Geometries, Cauchy, Riemann and Weirstrass gave mathematical rigour to the study of Calculus. Abel and Galois studied the solvability of various polynomial equations leading to the development of Group Theory. Cantor, Peano, Hilbert, Russel and Whitehead were responsible for the increasing use of logical reasoning in mathematics leading to a sound 'Mathematical Foundations.'

Twentieth century saw an explosion in the area of research in mathematics. Hilbert posed a list of 23 hitherto unsolved problems and these problems became the focus of Mathematical Research during the most part of the twentieth century. Many conjectures which remained unresolved for centuries were solved. For example, Kenneth Appel proved four colour theorem (1976); Andrew Wiles proved Fermat's Last Theorem (1995-1998) and Godel proved that Continuum Hypothesis is independent of standard axioms of set theory (1998). Many advancements were made in group theory, metric spaces, topological spaces, differential geometry and algebraic geometry during this century.

During the present century, the mathematical knowledge is growing in an exponential way.

1.9 Contributions of Indian Mathematicians

In India, mathematics was given supreme importance as a branch of knowledge ever since the Vedic period. Following sloka from Vedanga Jyotisa translated from Sanskrit is a testimony to this.

"Like the crests on the heads of peacocks, like the gems on the hoods of the cobras, Mathematics rests at the top of the vedanga shastras."

In this ancient text dating back to the Vedic period, we are introduced to irrational numbers like $\sqrt{2}$, $\sqrt{3}$ etc. Ancient sutras like Baudhayana and Apastambha sutras give very good rational approximations of the irrational number $\sqrt{2}$, which is correct to 5 decimal places. Even Pythagoras theorem finds place in those sutras though far more ancient than Pythagoras himself. Even the famous problem of 'squaring a circle' which remained unsolved for centuries was tackled by authors of sulba sutras.

Perhaps the most significant contribution of Ancient Indian Mathematics is its invention of zero, decimal representation of numbers and infinity. A list of numeral denominations in powers of 10 up to 10^{12} called 'parardha' is mentioned in the Ancient text 'Yajurveda Samhita' which is remarkable as the contemporary Greeks had knowledge of 10^4 as the highest power of 10. Ancient Indians had the knowledge of place values and even used symbol for zero by '.' while expressing a number in terms of digits using place values as used by 'Pingala' before 200 BC in his chandah-sutra.

Ancient Jain Texts as early as 500 BC to 200 BC, like 'Jambu dvipa Prajnapti' and 'Surya Prajnapti' have records of great mathematical achievements including $\sqrt{10}$ approximated correct to 13 decimal places.

Great Indian Mathematician Mahavira in 9th century AD gave the formula for ${}^{n}C_{r}$, the number of ways in which r objects can be choosen from a collection of n objects as $\frac{n!}{(n-r)!r!}$ in his work 'Ganitha Sara Sangraha'. However, the credit for this formula is wrongly given to Herigone of the 17th century.

As early as in the 5th century AD one of the greatest mathematician of India, Aryabhata I gave approximate value of π correct to 4 decimal places as 3.1416 mentioning that a circle of diameter 20,000 has circumference equal to 62832 and that it is only approximate suggesting that π which is the ratio of the circumference of a circle to its diameter is not rational. This is really remarkable since it is only in 18th century that Lamber could prove that π is irrational.

It was in 5th century AD that Aryabhata I, also contributed to algebra and astronomy with his amazing discoveries. It was in his 'Aryabhatiyam' that he has given tables for the trigonometric ratio sine for angles for 0° to 90° at regular intervals of $\frac{15}{4}$ degrees. Bhaskara I in 6th century AD in his famous commentary on Aryabhatiyam provides interesting geometrical treatment of algebraic formulae. In 7th century AD, Brahmagupta gave the well known formula to the area of a cyclic quadrilateral, though failing to mention explicitly that this formula is true only for cyclic quadrilaterals and not general quadrilaterals.

Both Brahmagupta as well as his predecessors have contributed a lot to the theory of Diophantine equations of the first and second degrees of the type $ax - by = c$ and $Nx^2 + 1 = y^2$, while a, b, c and N are constant integers. It was Bhaskara-II of 12th century AD, who improved the method of solution of equations of the form $Nx^2 + 1 = y^2$ by what is called Cakravala or cyclic method which he achieved much before the advent of French Mathematician Lagarange in 1736. It is worth mentioning here that a problem to solve $61x^2 + 1 = y^2$ in integers posed by French Mathematician Fermat in 1657 which was solved ultimately by another renowned mathematician Euler only in 1732, was in fact completely solved by Bhaskara II in 1150 itself in his 'Siddhanta Siromani' by his Cakravala method.

Calculus was invented and developed independently by two great Mathematicians Newton and Leibniz in the 18th century. However, much before their time, Bhaskar-II had introduced the concept of derivative of course not with much rigour. He also had the idea that the derivative vanishes at extreme values. Bhaskara-II is also famous for his popular text 'Lilavati'.

Though ancient Indian Mathematicians have contributed a lot to the development of mathematics, the contribution of Indian Mathematicians after the 12th century is not much note worthy until the appearance of the great Indian Mathematician Srinivasa Ramanujan (born 1887) in the early nineteenth century except for some notable contribution by Kerala Mathematician during a brief intervening period. The prodigy in Ramanujan who lived only for 32 years remarkably contributed to the theory of numbers, fractional differentiation, hypergeometric series, elliptic functions, elliptic integrals, Ramanujan π-function. Even after almost a century, mathematicians from all over the world are still working on many of the conjectures he had predicted.

To sum up, as Indians we are proud to say that the Indian contribution to the development of mathematics is very significant and every Indian should be proud of it.

1.10 Aesthetic Sense in Mathematics*

Working in mathematics gives aesthetic pleasure to the mathematicians. They see and experience beauty in certain aspects of mathematics as an art or as a creative work. An example of beauty in Methods of Mathematics is an elegant and simple proof of the Pythagorean theorem. Bertrand Russel, one of the greatest Mathematicians and Philosopher describes the beauty of Mathematics in the following words:

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of a sculpture, without appeal to any part of our weaker nature, without gorgeous trappings of paintings or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is a touch stone of the highest excellence, is to be found in Mathematics as purely as poetry."

What else can express the beauty of mathematics in a better way? Paul Erdos, an Hungarian Mathematician who devoted his whole life for research in mathematics publishing more than 1000 research papers said "Why are numbers beautiful? It is like asking why is Bethoren's Ninth Symphony beautiful. If you do not see why, someone can't tell you. I know numbers are beautiful. If they are not beautiful, nothing is." "No body can explain you the beauty of mathematics, you have to experience it yourself."

1.10.1 Beauty in Mathematical Methods

Mathematical proofs have beauty on their own. We describe a very pleasing method of proof as elegant. An elegant proof has the following characteristics:

1. Uses minimum number of assumptions or already established results.
2. Derives a result from apparently unrelated results.
3. Is based on new and original insights.
4. Is concise and clear.
5. Can be adopted in solving problems of similar kind.

In looking for an elegant proof, mathematicians try to find new and independent proofs for already established results. Pythagorean theorem is one of the theorems for which a great number of proofs have been given in search of a more and more elegant proof. Hungarian mathematician Paul Erdos who did not believe in the existence of God used to speak of an imaginary book in which the God has written down all the most beautiful mathematical proofs. When he came across a proof which he thought was the most beautiful, he exclaimed, "This one's from The Book."

* Ref : wikipedia, the free encyclopaedia

1.10.2 Beauty in Mathematical Results

Some times mathematical results establish a relation between two areas of specialisation in mathematics which seem to be totally unrelated. Such results are usually referred to as 'deep' results. G.H. Hardy, a mathematician who is responsible for world recognition of Ramanujan observes that beauty of mathematical results arise from an element of surprise. Euler's identity is often referred as a deep result and is called as one of the most remarkable formula in mathematics.

Euler's identity is the equality

$e^{i\pi} + 1 = 0$

where

e is the Euler's number, the base of natural logarithms,

i is the complex number whose square is – 1,

π is the ratio of the circumference of a circle to its diameter.

What is so remarkable about its mathematical beauty? In this equality each of the basic mathematical operations of addition, multiplication and exponentiation occur exactly once each. Further, the identity gives a relation between:

(i) the additive identity 0,

(ii) the multiplicative identity 1,

(iii) the number π which is predominant in trigonometry, Euclidean geometry and analytical mathematics,

(iv) the number e, the base of natural logarithms; and

(v) the number i, the imaginary unit of the complex numbers whose study leads to deeper insights into many areas of algebra and integral calculus.

All the five quantities which seem to be totally unrelated are closely related by this identity. In fact a poll of readers conducted by 'The Mathematical Intelligencer Magazine' named Euler's Identity as the 'most beautiful theorem in Mathematics' The 'surprise element' of Hardy here is how is it that all these seemingly unrelated quantities are connected by such an elegant formula.

1.10.3 Beauty in the Mathematical Language

The language of mathematics, the way of expressing mathematical results and proofs also fascinates mathematicians. Appreciating the language of mathematics, a great scientist Galileo said" mathematics is the language with which the God wrote the Universe."

1.10.4 Beauty in Mathematical Experience

Most mathematicians enjoy the mathematical beauty while actively engaged in mathematics. It is particularly in number theory that while manipulating with numbers, the mathematical experience becomes a delight. It is true with many other branches also. Bertrand Russel refers this as austere beauty of mathematics.

1.10.5 Three Aesthetic Experience Variables Identified by Birkhoff

According to George David Birkhoff, the typical aesthetic experience of an object is a function of three variables namely

(i) Complexity (C) of the object

(ii) The feeling value or aesthetic measure (M) and

(iii) Harmony, symmetry or order (O) of the object.

In terms of these variables, an aesthetic experience is a sequence of following phases:

1. A preliminary effort of attention which increases in proportion to the complexity of the object.
2. Feeling of aesthetic measure which awards this effort.
3. A realisation that the object is characterised by a certain harmony, symmetry or order which is concealed in the object.

An analysis of the aesthetic experience leads us to believe that the aesthetic feelings are because of an unusual degree of harmonious inter-relationship within the object. These elements of harmonious inter-relationship or order are repetition, similarity, contrast, equality, symmetry, balance and sequencing. These elements have positive effects on the aesthetic measure. But the complexity of the object makes it more difficult to experience the aesthetics of the object. Thus, aesthetic measure is given by the formula

$$M \,\alpha \frac{O}{C}.$$

1.10.6 Coexistence of Precision and Beauty in Mathematics

Mathematics is one place where both precision and beauty are always united. G.H. Hardy in his famous book 'A Mathematician's Apology' demonstrates that mathematics though precise, is initimately about beauty. He gives two examples to illustrate that beauty and precision coexist in mathematics. One example is 'Showing that there is no rational whose square is 2' and the other 'Showing that there is no largest prime number'. Both the proofs are easily graspable, but are precise and stir the soul with delightful wonder.

EXERCISE 1.6

(1) Give proof to the two examples given in 1.10.6.

1.11 The Scope of Mathematics

Mathematics first grew out of the need to do calculations in daily life and commercial activities, to measure land around and to predict astronomical events happening around us. We can broadly subdivide mathematics into the study of structure, space and change. The human quest for study of space leads to the development of geometry, first the Euclidean geometry and trigonometry of familiar two and three-dimensional and also of more dimensional spaces. Later, Euclidean geometry was also generalised to non-Euclidean geometries which play a central role in general relativity. The study of structure starts with numbers, first the familiar natural numbers and integers and their arithmetical operations, which leads to elementary algebra. The deeper properties of whole numbers are studied in number theory. The investigation of methods to solve simple equations leads first to the invention of rational numbers, real numbers and complex numbers. More investigations into properties of these numbers and attempts to find methods to solve equations lead to the development of the field of abstract algebra, which, among other things, studies rings and fields, structures that generalise the properties possessed by these numbers. Long standing questions about ruler-and-compass constructions were finally settled by Galois Theory a branch of abstract algebra. The study of physically important concept of vectors, and their structural properties lead to definitions of vector spaces and the study of linear algebra. This branch of mathematics relates to the study of structure and space. The modern fields of differential geometry and algebraic geometry generalise geometry in different directions: differential geometry emphasises the concepts of functions, fiber bundles, derivatives, smoothness, and direction, while in algebraic geometry geometrical objects are described as solution sets of polynomial equations. Group theory investigates the concept of symmetry abstractly; topology, the greatest growth area in the twentieth century, has a focus on the concept of continuity. Both the theory of Lie groups and group theory reveal the intimate connections of space, structure and change.

Calculus was developed as a most useful tool for the understanding and describing change in measurable quantities which is the common theme of the natural sciences. The concept of function was introduced as a central concept to describe a changing variable. Finding methods to solve problems dealing with the relationships between a quantity and its rate of change, lead to the study of differential equations. The numbers used to represent continuous quantities are the real numbers, and the detailed study of their properties and the properties of real-valued functions is known as real analysis. For several reasons, it is convenient to generalise to the complex numbers. Complex numbers, their properties, functions of complex variables and their properties are studied in complex analysis. Functional analysis is the study of (typically infinite-dimensional) spaces of functions, which laid groundwork for the study of quantum mechanics. The study of many phenomena in nature lead to study of dynamical systems, investigations into the precise ways in which many of these systems exhibit unpredictable yet deterministic behaviour lead to the development of chaos theory, The fields of first set theory to start with and then mathematical logic were developed in order to clarify the foundations of mathematics. Mathematical logic, which comprises of recursion theory, model theory and proof theory, is now closely linked to computer science.

When electronic computers were first conceived, several essential theoretical concepts were developed by mathematicians, leading to the computability theory, computational complexity-theory, and information-theory. Many of those topics are now investigated under theoretical computer science. The branches of mathematics that are most generally useful in computer science are commonly referred to now as Discrete Mathematics. The mathematics of chances or the study of uncertainty in certain terms is the mathematical theory of probability which allows the description, analysis and prediction of phenomena where chance plays a part. Statistics is an important field of applied mathematics, which uses probability theory as a tool, and it is used in all sciences as well as social sciences. Numerical analysis is the branch of mathematics, which is the study of methods for efficiently solving a broad range of mathematical problems numerically on computers, which are otherwise beyond closed solutions, to obtain credible answers.

This broadly is the scope of mathematics in general.

1.11.1 Scope of Secondary School Mathematics Education

It is at the Secondary School that the student begins to come across mathematics as an academic discipline. Till the end of upper primary stage, mathematics education is guided by the logic of children's psychology of learning mathematics rather than the logic of mathematics itself. It is at the secondary school that learning becomes more logical and the student starts understanding the structure of mathematics. So the notion of 'argumentation' and 'mathematical proof' becomes the focal point of mathematical learning. She/He starts using rigorous and highly stylised mathematical terminology. She/He becomes comfortable in the use of language of mathematics with carefully defined terms and concepts, use of symbols, precisely stating 'propositions' using only already defined terms/concepts, giving proofs for justifying propositions and proving more propositions and 'theorems' with the help of already established propositions. It is through the study of geometry that the student starts appreciating the structure of mathematics viz, mathematical terms, propositions, proofs and theorems. A substantial part of secondary school mathematics is devoted for consolidation of mathematical ideas introduced earlier. Students start reasoning about the shapes and the formulae, they have been associating with these shapes during in their primary classes. Students study here more rigorously the elementary algebra introduced to them earlier at the upper primary stage. They become familiar with uses of algebraic manipulative techniques at many places for proofs in geometry.

However, it is also important for the student to visualise geometrically what she/he has proved using algebraic machinery. Problem solving ability is developed in the student during the secondary stage. For this, she/he integrates many mathematical techniques which she/he has already come across into a problem solving ability. Problems which involve more than one area are posed to the students. Mathematics used in physical, biological and social sciences are used to inspire the students immensely. Mathematical modelling, data analysis and interpretation are introduced to them at this stage. For instance, for an environmental related problem he may have to set a mathematical model, solve it, visualise the solution, and then deduce a property of the modelled system.

Summary

Mathematics has its origin in daily life needs and happenings in the world around us. Many branches of mathematics emerged out of sheer necessity to solve problems in physical, biological and social sciences. In this Unit, we have discussed the origin, nature and development of mathematics as a discipline.

We also discussed the building blocks of mathematics namely, undefined terms, definitions, axioms, propositions, open sentences, and theorems. There are two kinds of propositions – prime and composite. Composite propositions are got by combining propositions using sentential connectives. Different sentential connectives are conjunction, disjunction, implication, negation, and equivalence. We discussed the truth values of propositions and wrote truth tables for composite propositions. We introduced arguments and using the truth tables, determined the validity of arguments.

We then discussed the calculus of open sentences and their Venn diagrams. We defined a mathematical theorem and discussed its invariants namely, converse, inverse and contrapositive. We introduced proof of a mathematical theorem and discussed different types of proofs. We also saw distinction between a proof and verification.

We then briefly reviewed history of world mathematics starting from prehistoric period to most recent modern period. We appreciated the great contributions of Indian mathematicians to the world of mathematics. We ended the Unit with a brief discussion on the scope of mathematics in general and the scope of secondary stage mathematics education in particular.

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UNIT 2

EXPLORING LEARNERS

2.1 Introduction

Learning is a continuous and commulative process which starts from conception. In fact, human beings are natural learners showing a continuous desire to learn and curiosity for discovering meaning in reality and to provide the World with its meaningful interpretation. This natural tendency is not limited to real-life situations outside the classroom, but prevails inside the classroom also when students deal with learning materials. The belief that child comes to school as 'blank slate' is no more tenable. The child comes to school with certain social, moral, emotional and intellectual background with his/her own understanding of the phenomena around him/her. The teacher's role is to explore social, moral, emotional and intellectual background of the learners to create better teaching-learning environment.

To 'explore' means to 'search through' or to 'look into closely'. Thus, exploring learners means to examine or scrutinise learners in terms of their 'readiness' to learn in general and mathematics in particular. In the context of learning mathematics, it signifies assessing learners prior to instruction in terms of prerequisite knowledge and skills, general intellectual abilities, specific mathematical aptitude and degree of motivation, necessary for learning the new content. Exploring learners helps a teacher to develop a sound pedagogical foundation and plan his instructional strategies in such a way that expected learning outcomes are maximised. It also signifies assessing learner's field of experiences and inner potentialities. Cultivation of skills and their nurturance along with provision of extending opportunities are the most significant aspects of imparting education. Exploring process entails development of scientific temper, right attitude, systematic way of assimilating knowledge, identification of difficulties, confronting the process of understanding and their remediation. It facilitates the process of teaching and learning by stimulating and sustaining learners interest through probing, prompting for different ways of thinking, raising queries and motivating the process of teaching and learning.

In this Unit, we will discuss some important components of exploring processes and illustrate them with examples. Significant aspects of the exploring process encompass skills of listening, probing, raising queries and dialogue among members of peer group. Harmonious blending of these results in building and promoting confidence in learners. A professionally competent mathematics teacher has to follow certain general principles in exploring learners. These principles are discussed in the following sections.

Learning Objectives

After studying this Unit, the student-teachers will be able to

- relate the subject matter to the student's field of experience and intuition
- encourage the student for probing, raising queries and promoting dialogue among peer-group
- promote student's confidence in learning mathematics and to apply it in day-to-day life.

2.2 Assessing Students' Readiness

All achievements in human life depend on three major factors - ability, opportunity and motivation. These three factors define an individual's readiness to achieve what he/she wants to. At the first place, an aspiring person should possess necessary abilities – physical and intellectual; secondly, he should have rich experiences and learning opportunities in the relevant field of activity. These experiences function as starting points for any further accomplishments, and thirdly, one should have requisite degree of motivation to achieve his/her goal. In the learning of a subject like mathematics, children build on the existing foundation in terms of previous knowledge and thinking. Building on students' initial level of thinking, teachers can plan mathematics learning experiences in such a way so as to draw the most out of their existing proficiencies, interests and experiences. While planning lessons, effective teachers place the current knowledge and interests of the students at the focus of their instructional strategies. In this context, ongoing assessment of students' competencies, including language proficiency, reading and listening skills, ability to cope with complexity, and mathematical reasoning, is essential. In addition, it is also desirable to assess students' general mental ability, numerical aptitude, attitude towards mathematics as a subject of study and degree of motivation to participate in mathematical learning. The results of such an assessment may help teachers adjust their instructions to meet the learning needs of students. Many informal ways of assessing learners are discussed in Unit 8.

Research has shown that building on students' existing capabilities and motivational status is a better pedagogical approach than trying to make up for existing gaps and weaknesses through remedial teaching. Teachers should understand that there are multiple reasons for students' deficiencies in learning. Teachers should organise discussions with small groups of students or with the whole class, focussing their attention on common errors and difficulties, so as to help them learn from their own errors. Mistakes offer opportunities to examine errors in reasoning, and thereby raise everyone's level of analysis. Mistakes are not to be covered up; they are to be used constructively. In the process, students share a variety of interpretations and strategies which enable them to re-evaluate their own ideas in comparison to those of others. Effective teachers always start from the point where their students initially are. They identify the existing level of learner's understanding and readiness. Accordingly, teacher must set instructional objectives and appropriate content matter to maximise learning opportunities for learners.

This helps them design their teaching strategies and appropriate levels of challenging problems for their students. At the same time, teachers can also find out ways and means to help academically weak students by reducing the complexity and difficulty levels of learning tasks. In order to increase the challenge posed by teaching tasks in all classrooms, teachers can put some gaps in the solutions of problems, requiring students to fill up these gaps with reasoning and generalisations.

2.3 Relating Learning of Mathematics to Learners Real-life Situations

The first and foremost responsibility of a mathematics teacher in the classroom is to help the child construct for himself a proper meaning of mathematics as a subject of learning as well as an effective instrument to deal with social, economic and psychological problems frequently encountered in daily life. For example, suppose, teacher wants to introduce the topic trigonometry. He/she can introduce it in different ways. One way is to start directly by using the different measures of a right angle triangle and by talking about different ratios based on sides, etc. Another way, he/she can start it by asking the student about triangles, different sides of a triangle, etc. (checking prerequisites). For checking prerequisites, teacher can pose a simple puzzle related with height and distance [e.g.– A tree lies exactly on the other bank of a river of width 50 m. Then can you measure the height of the tree without going to the other bank?]. Students will be more motivated and active in the teaching learning process. If the teacher uses the later way for introducing the topic. The teacher may provide, in a lesson, learning tasks which show a relation of the subject to the daily life of the child.

Meaning does not have to be essentially conscious, rather, can be constructed implicitly without making it dominant in the situation. Of course, every child has to construct his or her own meaning, but still it depends on the meaning given by teachers, parents and society as well as, on the students' personal experiences, abilities, dispositions, their wishes and intentions. Learning of a subject like mathematics cannot be a joyful experience for a learner, if it is carried out with the help of memory based repetitive exercises. Such an approach, on the one hand, makes the subject matter drab and boring, and on the other hand, places unnecessary load on memory faculties of the child.

Mathematics teachers should use tasks that allow students to develop their own strategies for solving problems and construct meanings for mathematical tools. Teacher may invite some working professionals to discuss with school students the ways in which they use mathematics in their professional lives, emphasising topics beyond arithmetic and including mathematics, topics in classes that relate to occupations. In this process, the immediate concern of a teacher is to involve the learner to think, to observe, to analyse and to reason through brain storming process and thereby sensitise them to cultivate scientific temper and right attitude towards mathematics. If learners are not properly motivated, they become indifferent and do not listen to what is being taught, rather, they become mentally absent, though physically they may be present in the class. Thus, there arises a basic question for teachers as to what needs to be done to improve their pedagogical approach, so as to cultivate learner's sensitivity towards learning.

Some of the effective measures for cultivating learner's sensitivity towards learning are:

(i) To relate the subject matter to the learner's field of experience

(ii) To enquire about the previously acquired knowledge in relation to what is being taught

(iii) To concretise teaching with help of teaching aids/activities or illustrative examples, so that learners may have the feeling of acquired knowledge as their own making, not as something thrust upon them

(iv) To make learners acquainted with application of concepts, more preferably, in daily life situations and also their relationship with other allied subjects

(v) To cultivate skill of appreciating, pedagogy has to depend upon the process rather than the product.

For example, if a teacher has to teach identity $(a + b)^2 = a^2 + 2ab + b^2$, merely stating the result goes either unheard by learners or crammed by them, because they do not have insight into underlying principles. So, they have no sense of appreciating the same. For the sake of cultivating appreciating skill, the pedagogical approach for introducing this result may be as follows:

The teacher may begin with concrete examples for generating the pattern such as:

$(1 + 2)^2 = (1 + 2)(1 + 2) = 1 \times 1 + 1 \times 2 + 2 \times 1 + 2 \times 2 = 1^2 + 2(1 \times 2) + 2^2$,

$(3 + 4)^2 = (3 + 4)(3 + 4) = 3 \times 3 + 3 \times 4 + 4 \times 3 + 4 \times 4 = 3^2 + 2(3 \times 4) + 4^2$, and so on.

The teacher may add a few more similar examples, including negative numbers or expressions involving zeroes and motivate students to look at the inherent pattern and find its generalised form, i.e., for any real numbers a and b; $(a + b)^2 = a^2 + 2ab + b^2$.

The teacher may organise a simple activity to provide insight into the identity and to evoke learner's appreciation. He/she may demonstrate atleast one activity ensuring active involvement of learners and then asking them to perform similar activities. In this way, the role of a teacher has to be like a facilitator. For instance, the teacher may initiate the process of demonstration as follows:

Take a = 10 cm, b = 3 cm. On a thick sheet of paper, construct a square having its side 10 cm, another square having each side 3 cm and two rectangles each having length 10 cm and breadth 3 cm each and make cut outs of these shapes. Arrange all these together giving a larger square having each side 13 cm. Calculating area, we get

$(10 + 3)^2 = 10^2 + 2 \times 10 \times 3 + 3^2$ as shown in the Fig. 2.1

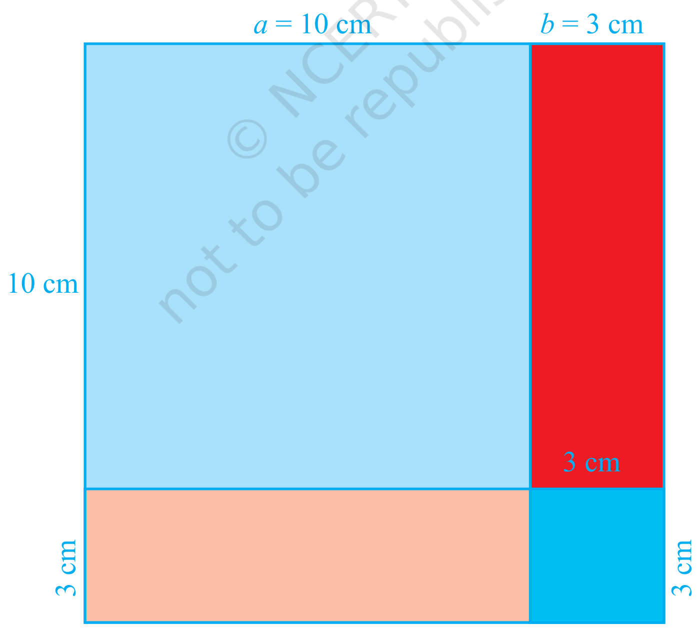


Fig. 2.1

Thus, systematic presentation of abstract concept get concretised and learners appreciate and assimilate the result. Such a process cultivates learning skill by stimulating learner's interest, thus ensuring their massive participation. Even after such rigorous pedagogical way of dealing with content, there may be some students who still remain unattentive and demotivated. To overcome this problem, teacher needs to bring change in pedagogy by identifying such lot of learners and adopt remedial teaching separately until and unless they find themselves confident and cultivate learning skill. Note that while performing this activity, a teacher is exploring the learner's status of learning time to time, and accordingly, changing his/her teaching strategy. Sometimes, puzzles, stories and game activities based on theme/ concept help in cultivating learning skill in learners.

2.4 Allowing Students to Attempt Problems Independently

Most of the times, it is difficult for children to grasp a new concept or solve a problem when distracted by the views of others. Therefore, teachers of mathematics should ensure that students are given requisite opportunities to think and work quietly by themselves, so that they are not required to process the varied, and sometimes conflicting perspectives of other students. Research has established that while making sense of ideas, students need opportunities to work both independently and collaboratively. At other times, they need to be in pairs or small groups, so that they can share ideas and learn from each other. Also, there are times when they need to participate actively in purposeful class discussions, where they could get opportunities to clarify their understanding and be exposed to broader interpretations of the mathematical ideas under focus. Such opportunities will provide learners broader perspectives and better understanding of already existing conceptual knowledge. Moreover, such strategies help promote confidence among students.

The regular classroom discussions should form a part of routine teaching–learning activities. Such discussions can play a significant role in the development of students' mathematical thinking and imbibe in them a favourable attitude to mathematics as a field of knowledge. Effective teachers should ensure that all students are provided with opportunities to struggle with mathematical problems independently. Teacher may offer assistance as and when students required. Students' participation may be ensured by encouraging questions directed towards the solution of the problem, and the responses that students contribute may be used in arriving at the solutions to mathematical problems under discussion. Students need time to think and work on the problems quietly by themselves, without any intervention of other peers. But, sometimes, group discussion with partners or peers provides useful guidelines for arriving at correct solutions to tedious and tricky mathematical problems. The teacher may form small learning groups in the class with students of varying levels of abilities and competencies included in each group. Such arrangements are useful not only for enhancing engagement, but also for exchanging and testing ideas and generating higher level of thinking, and thus, are responsive to their learning needs. In such situations, students learn to make conjectures and formulate hypotheses regarding probable solutions to problems by engaging in critical argumentation and validation of information. When learning groups are mixed in terms of ability, academic achievement and motivation, new insights are developed at varying levels, which lead to enhanced overall understandings. The function of teachers is to clarify in advance the expectations and role of participation to every member of the group and ensure that roles, such as listening, writing, answering, questioning, and critically assessing, are understood and properly discharged. In whole class discussion, teachers are the primary resource for nurturing patterns of mathematical reasoning. Teachers manage, facilitate, and monitor student participation and they record students' solutions, emphasising efficient ways of doing this. While ensuring that discussion retains its focus, teachers invite students to explain their solutions to others; they also encourage students to listen and to respect one another, accept and evaluate different view points, and engage in an exchange of thinking and perspectives. Note that this way, a teacher is exploring the strength and weakness of learners at the time when the student is doing the problem independently explaining his/her solutions to others, discussing the problem in groups.

The classroom discussion strategy may be based on the techniques of probing, questioning and through these devices, learners should be motivated to grasp the knowledge being imparted. In mathematics, pedagogy should dwell upon two important aspects of problem solving, i.e., what are the given data and what has to be worked out. In between these two extremes, lie mathematical manipulations involving axioms, premises, reasoning, deduction, analysis and synthesis.

Let us illustrate it by an example:

Example: Find out the area enclosed within a circle of radius *r* units. To help children in solving this problems the teacher should begin with queries as follows:

1. What is given?
2. What is to be found?
3. What do you mean by the area enclosed within a circle?

After having made a few such basic queries, pedagogy should be changed from probing to solving, i.e., from 'what' to 'how'? Teacher should motivate students to probe the situation enabling them to suggest different ways as to how the region enclosed by the circle can be divided into finite number of small regions which altogether constitute the area of the circle.

In mathematics, probing also has its limitation in the sense that there are certain statements which have been assumed to be always true needing no proof and verification. However, they are intuitively clear and come under the purview of cognisance through observation and experience. For example, "Two lines in a plane are either parallel or intersecting;" "Equals have equal halves;" "The whole is greater than its part" etc. These contexts should be taken as learning opportunities to experiment and verify mathematical premises.

2.5 Encouraging Cooperative Learning Strategies

Classroom should be considered as a learning community. It has been observed through experience that students want to learn in a collective environment in which a feeling of togetherness is generated and fostered. It is, therefore, teacher's responsibility to see that everyone feels included by respecting and valuing the knowledge of mathematics and the cultures that students bring to the classrooms. This ensures a feeling of security and allows every student to get involved. However, it is important that the caring relationships so developed do not encourage students to become overly dependent on teachers or fellow students. Effective teachers promote classroom relationships that allow students to think for themselves and ask questions. They have high yet realistic expectations about enhancing students' capacity to think, reason, communicate, reflect upon and criticise their own practice, and provide students with opportunities to ask why the class is doing certain things and with what effect. The relationships that develop in the classroom become a resource for developing students' mathematical competencies and identities.

The classroom discussion can provide a useful platform for broader interpretations and an opportunity for students to clarify their misunderstandings. It can also assist students in solving challenging problems when a solution is not initially available. Teachers have an important role to play in the discussion. Focussing attention on efficient ways of recording, they invite students to listen and respect one another's solutions and evaluate different view points. In all forms of classroom organisation, it is the teacher's task to listen, to monitor how often students contribute, and to keep the discussion focussed. When class discussion is an integral part of an overall strategy for teaching and learning, students provide their teachers with information about what they know and what they still need to learn.

There are many ways in which teachers can provide opportunities for students to learn from their errors. One is to organise discussion that focus student's attention on difficulties that have surfaced. Another is to ask students to share their interpretations or solution strategies, so that they can compare and re-evaluate their thinking. Yet another is to pose questions of higher order thinking skill that are needed to be resolved. Open-ended questions can provide scope of divergent thinking and multiple processes.

Teacher should not reveal everything to students, rather, let the students guess before he/she tells it, i.e., let the students find themselves as much as feasible. While exploring learners, teacher should avoid negative comments. For example, suppose a student commits mistake in certain long computation which goes through several lines, teacher should refrain from saying directly that his computation was wrong, rather teacher should prefer to go through the computation with student, line by line and say, "You began correctly. Your first step and second step are correct, but what do you think about the third step?" The mistake is in third step and if the student discovers it by himself/herself, then he/she has a chance to learn something. Instead, if teacher says "this is wrong," then student feel offended and then he/she will not listen to anything and lose interest in the subject and if such incidents are repeated frequently, then student may develop attitude of indifference and start disliking mathematics.

Creating a positive classroom environment is a major responsibility of teachers. It requires that both teachers and students have an active role in discourse, particularly problem solving, questioning, listening, clarifying, and problem posing. The teacher has the responsibility to recognise students' needs while facilitating possible pedagogy, i.e., what students learn is fundamentally connected with how they learn it. What students learn about particular concepts and procedures as well as about thinking mathematically depends on the ways in which they are engaged in mathematical activities in their classrooms. If teacher is introducing students to a new manipulative material, it is important to allow them time for free play and exploration. It is truly a natural process that satisfies human curiosity. This may be accomplished by involving the class in a brief discussion of what they learn. The characteristics of the learner and the nature of the content are chief considerations in providing conducive learning environment. Learning environment should enhance children's problem solving skills which invite children to initiate questions and problems, make and investigate conjectures, listen to and question the teacher. This way, students may become more independent, and confident learners of mathematics. The role of the teacher should, therefore, shift from that of the sole conveyer of knowledge to one who facilitates, guides and helps students in their mathematical explorations.

For efficient learning, the learner should discover by himself/herself the material to be learnt as feasible under the given circumstances. For efficient learning, the learner should be interested in the material to be learnt and find pleasure in the activity of learning. Learning needs to be conducted through three phases, i.e., the phases of exploration, formalisation and assimilation. Exploratory phase is closer to action and perception and moves on a more intuitive and more heuristic level. The second phase of formalisation corresponds to conceptual level, introducing terminology and proofs. The third phase of assimilation implies that the material learnt should be mentally digested. Thus, for efficient learning, exploratory phase should precede the phases of formalisation and assimilation. These phases seek to connect the topic to be learnt with World around us and with other knowledge.

In order that teaching by one should result in learning by the other, there must be some sort of contact or connection between teacher and the student. The teacher should be able to see the students' understanding, expectations and difficulties. The responses of the students to teaching depend on their background, their outlook and their interest. Therefore, teacher should keep in mind and take into account, what students know and what they do not know.

At any rate, teacher has to explore his students in terms of their difficulties and diagnose accordingly.

Many cooperative learning strategies to encourage and motivate learners are discussed in Unit 9.

2.6 Promoting Classroom Dialogue

The ways of communicating mathematically demands skilful work on the part of the teacher. Students need to be taught how to articulate sound mathematical explorations and how to justify their solutions. Students and teacher both need to listen and debate to others' ideas and to use debate to establish common understandings. Listening attentively to students' ideas helps teachers to determine when to step in and out of the discussion, when to press for understanding, when to resolve competing students' claims, and when to address misunderstandings or confusions.

For this purpose, dialogue process is very useful. Dialogue can be conducted in many ways. Some of the suggested ways are as follows:

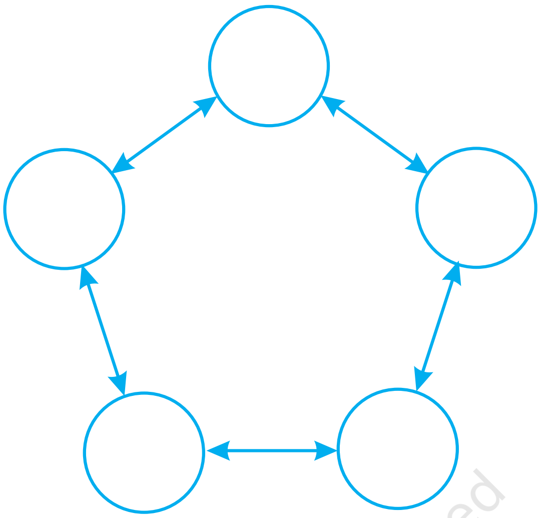
- The teacher announces the topic in advance and asks learners to prepare the topic for presentation by individual learner asking others for interactive discussion.
- Teacher forms different groups of learners and arranges dialogue among the groups on the assigned topics.
- Teacher announces the topic and initiates the discussion involving students through probing and questioning.

Different Approaches to Engage in Dialogue

One-to-one Pairing

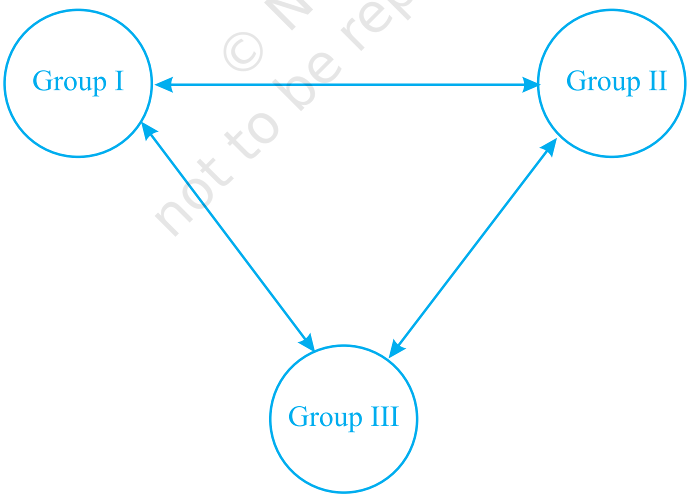
(Teacher observes as facilitator)

Within Group



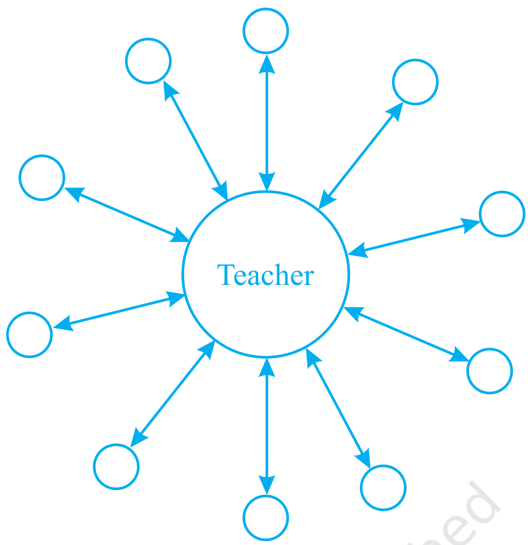
(Teacher observes as facilitator)

Among Groups



(Teacher observes as facilitator)

Teacher as Initiator



(Teacher and entire class)

In dialogue approach, the problem is tackled in interactive way, through questioning and queries as exemplified below:

1. TEACHER: Today, we will discuss addition of irrational numbers.

Teacher initiates the dialogue by posing the problem: "Is sum of two irrational numbers always an irrational number"?

MEENA: Yes, sir! It is always an irrational number.

Entire class investigates the response of Meena. After few minutes, another student reacts to Meena's response.

SURESH: No, Meena! It is not always true.

He selects a pair of irrational numbers $1 + \sqrt{2}$ and $1 - \sqrt{2}$ and shows that its sum is 2 which is not an irrational number.

TEACHER: Can anyone else choose such a pair of irrational numbers?

ASLAM: Yes, sir! Take a pair of irrational numbers as $\sqrt{2}$ and $-\sqrt{2}$. Their sum is 0, which is also not an irrational number.

TEACHER: Yes, very good. Now, can you tell me how many such pairs of irrational numbers are possible? Students discuss among themselves and finally one of the students responds as follows:

RAGHAV: Sir, I have found 8 such pairs. I think there may be some more also.

GEORGE: Sir, there may be infinitely many such pairs of irrational numbers.

TEACHER: Is there any student who does not agree with what George has stated ?

Again entire class investigates by making different cases and no one contradicts George's response. Then, finally teacher concludes the discussion by making factual statement, i.e., sum of a pair of irrational numbers may or may not be an irrational number.

Let us illustrate by another example as to how dialogue approach can be applied in problem solving.

2. Teacher Poses the Problem

A sum of Rs 20000 is borrowed by Hina for 2 years at an interest rate of 8% per annum compounded annually. Find the compound interest (CI) and the amount she has to pay at the end of 2 years.

ASLAM: Madam! should we find the interest year by year? If so, then how?

TEACHER: Yes, you may do it this way:

Step 1: Find the simple interest (SI) for one year. Let the principal for the first year be P_1. Here

P_1 = Rs 20000

$$SI_1 = \text{SI at 8\% per annum for the first year} = \text{Rs } \frac{20000 \times 8}{100} = \text{Rs}1600$$

Step 2: Amount at the end of the first year = $P_1 + SI_1$

= Rs 20000 + Rs 1600 = Rs 21600.

So, P_2 = Rs 21600, where P_2 denotes the principal for the second year.

Step 3: Find the interest on this sum for one year. Thus,

$$SI_2 = \text{Rs}\frac{21600 \times 8}{100} = \text{Rs}1728$$

Step 4: Find the total amount which Hina has to pay at the end of second year.

Amount at the end of 2^{nd} year

= $P_2 + SI_2$ = Rs 21600 + Rs 1728 = Rs 23328

Total interest to be paid = Rs 1600 + Rs 1728 = Rs 3328

Reeta: Madam! Is this interest (Rs 3328) different from SI calculated on Rs 20000 for two years ?

Teacher: Why don't you find SI for 2 years by yourself?

Reeta: Yes, madam!

She calculates it as follows:

$$\text{SI for 2 years} = \text{Rs } \frac{20000 \times 8 \times 2}{100} = \text{Rs } 3200$$

She says that CI is different from SI for two years.

Zubeda: Madam! Here, the number of years is 2. What will happen if number of years is 4 or 5 or 6?

Teacher: We can again proceed in the same way as explained earlier.

Zubeda: Madam! Can we do it in a shorter way?

Teacher: Yes, there is a shorter method. We can use the following formula for calculating CI.

i.e., $$\text{CI} = \text{P}\left(1 + \frac{r}{100}\right)^n - \text{P}$$

where P is the principal, r is the interest rate % per annum and n is the number of years.

The students find CI using this formula.

Apoorva: Madam! I have calculated CI to be Rs 3328 by using the above formula.

Teacher: Wonderful!

Rekha: Madam! How did you find this formula?

Teacher now explains derivation of the formula.

Students appreciate the easier and shorter method for finding the amount compounded annually and teacher further generalises the procedure for situations when the interest is compounded half yearly or quarterly using the idea of conversion periods.

3. Teacher asks students to solve the inequality

$(x – 3) (x – 5) > 0$

Hamida: Sir! Since $x – 3 > 0$, and $x – 5 > 0$, so $x > 5$ is the solution.

Teacher: Harish! Have you also solved it?

Harish: Yes, Sir! I have solved it as follows:

$x – 3 > 0$, and $x – 5 > 0$, so $x – 3 > 0$

i.e., $x > 3$

RUHI: When $x = 4$, we have $x - 3 = 4 - 3 = 1$, $x - 5 = 4 - 5 = -1$

So, $(1) \times (-1) = -1 < 0$

So, $x = 4$ does not satisfy the inequality. Therefore, $x > 3$ will not be a solution to the given inequality.

TEACHER: It means the solution $x > 3$ given by Harish is not correct.

Teacher now again asks other students. Is the solution $x > 5$ given by Hamida correct?

ANU: Yes, Sir! I have verified it by taking different values of x greater than 5.

TEACHER: Is it the only solution?

ARUNA: Sir! $x = 2$ is also a solution.

SINGHVI: Sir! $x = 0$ also satisfies the inequality. So, $x = 0$ is also a solution.

RAMESH: Sir! all values of $x < 3$ satisfy the given inequality.

TEACHER: $x < 3$ and $x > 5$ are the solutions of the given inequality and explains how these solutions have come up.

4. Teacher poses a question: Prove that a diagonal of a parallelogram divides it into two congruent triangles.

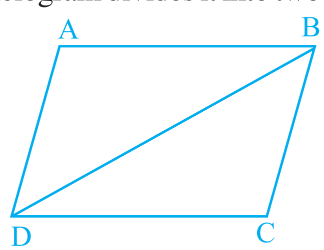


Fig. 2.2

STUDENT (1): I can prove it.

TEACHER: Which parallelogram are you taking?

STUDENT (1): Parallelogram ABCD.

Then he draws parallelogram and a diagonal DB

He proves it like this – In ΔADB and ΔCBD,

DB = DB (common)

AD = BC [opposite sides of a parallelogram]

AB = DC (opposite sides of a parallelogram)

So, Δ ADB ≅ Δ CDB (SSS).

TEACHER: Is this proof correct ?

STUDENT (2): No!

TEACHER: Why?

STUDENT (2): Correct proof is as shown below:

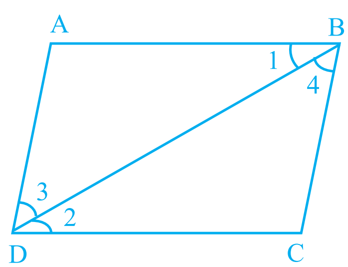


Fig. 2.3

In ΔABD and ΔCDB,

∠ 1 = ∠ 2 (alternate angles)

$\angle 3 = \angle 4$ (alternate angles)

DB = BD (common)

So $\Delta ABD \cong \Delta CDB$ (ASA).

Teacher: What is wrong with the first proof?

Student (2): In the first proof, we have used the property that opposite sides of a parallelogram are equal,

i.e., AD = BC and AB = DC.

Teacher: So what, what is wrong?

Student: It is based on the congruency of the triangles ADB and CBD itself. So, we cannot use AD = BC and AB = DC to prove $\Delta ADB \cong \Delta CBD$.

Teacher: It means that if we use statement p to prove statement q, then we cannot use statement q to prove statement p. Is it?

Students: Yes Sir !

Teacher: Do you know, it is called circular reasoning?

Students: 'Oh!'

Teacher: Is the second proof given by Student (2) correct?

Student (3): Yes, Sir!

He has used the basic property of the parallelogram (opposite sides are parallel) to show $\angle 1 = \angle 2$, and

$\angle 3 = \angle 4$ (alternate angles).

Teacher: Very Good.

In fact, the second proof is correct. Thus, *for a correct proof, we should not use the result (1) to prove another result (2) which itself is based on result (2).*

Thus, pedagogy must dwell upon resources of creativity and exploration. Teaching should be in the conversational mode rather than traditional authoritarian way. The conversational way helps children to grow in self confidence and awareness and relating teaching with his/her own experiences and logically constructed arguments.

As stated earlier, teaching of mathematics needs to be conducted through exploratory phases culminating into formalism, analysis and synthesis leading to problem solving. At each step of teaching, teacher has to explore his learner while exploring the process of mathematisation. Probing and questioning are essential elements of exploratory process since teaching has to be conducted in accordance with understanding, inner potentiality and mental ability/capacity of the learner. Let us elaborate upon these with one more example on problem solving.

Problem: A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2m and volume is $8m^3$. If building of tank costs Rs 70 per square metre for the base and Rs 45 per square metre for sides, what is the cost of least expensive tank?

Problem Solving: To solve this problem, the readiness of learner in terms of basic principles of maxima/minima and derivative need to be enquired through pertinent queries and questioning. The next approach of the teacher should be towards helping students in formulating the problem so as to apply the principles of maxima/minima.

Formulating the Problem

Ask students to conceptualise the shape of tank and draw rough figure of the open tank by assigning variable x (in metres) to the length and variable y to the breadth of the rectangular base of the tank. Further, symbolise the cost for the construction of the total rectangular faces of the tank by the symbol C. Now, ask students to record the given data as follows:

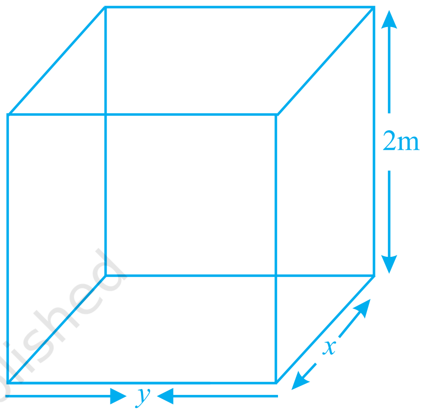


Fig.2.4

Volume of the open tank = $8m^3$

Depth of the tank = 2m

Cost of the rectangular base of the tank = Rs 70 per sq. metre

Cost of other four rectangular faces = Rs 45 per sq. metre

Teacher helps students to formulate the problem, i.e., volume of the tank = 2 × rectangular area of the base

$$8 = 2xy$$

i.e., $$xy = 4 \quad (1)$$

Total cost of construction $$C = 70\,xy + 180\,(x + y) \quad (2)$$

Solving the Problem

Teacher asks students, how to minimise the cost C? Teacher may recall that to find the desired result and to apply principle of maxima and minima, C has to be converted in to a single variable. Students, with the help of equations (1) and (2) obtain an expression for C in a single variable, i.e.,

$$C = 70x\left(\frac{4}{x}\right) + 180\left(x + \frac{4}{x}\right)$$

i.e., $$C = 280 + 180\left(x + \frac{4}{x}\right) \quad (3)$$

The teacher, further recalls the principle of maxima/minima and helps students to find the first derivative of C with respect to x and equates it to zero.

i.e., $$\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right) = 0,$$

which gives

$x = \pm 2$ (–2 is not admissible, ask students why ?)

Again, motivate learners for the next step and ask them to confirm that $x = 2$ is the point of minima by looking at the sign of $\frac{d^2C}{dx^2}$ which is positive for $x = 2$.

Finally, help learners to determine the minimum cost by putting the value $x = 2$ in equation (3), i.e.,

$$C = 280 + 180 \times 4$$
$$= 1000$$

Thus, it is obtained that the least cost of the tank is Rs 1000.

From above discussions, it may be inferred that exploring learner entails the procedure of investigating into the knowledge, understanding, abilities of symbolising, formalism, analysing and manipulative skills of learner. Thus, exploring learners and exploring mathematical procedures are closely related.

EXERCISE 2.1

1. What do you mean by the term 'exploring learners'? Discuss general principles in exploring learners.
2. What are the effective measures for cultivating learners' sensitivity towards learning? Illustrate with mathematical examples.
3. Discuss with mathematical examples the roles of probing, questioning and cooperative learning strategies while exploring the learners.
4. Discuss with mathematical examples based on dialogue approach for efficient learning of mathematics.

5. Elaborate upon the following phrases:
   (i) Relating mathematics to children's real-life situations/local context.
   (ii) Allowing learners to attempt problems independently.
   (iii) Encouraging Cooperative Learning Strategies

Summary

Exploring process entails development of scientific temper through the sequence of probings, promptings and dialogues between learners and teacher as well as among peer groups. Exploring means splitting the problem into pieces and then integrating the bits of information into the whole body of knowledge either for a particular concept or for solving a mathematical problem. Exploring also provide scope of multiple ways to communicate a mathematical idea, alternate ways to solve a problem and to bring new insight in learners' existing mathematical world.

Analysis, synthesis, logical reasoning, induction and deduction are the hallmarks of exploring process constituting the essence of pedagogy of mathematics.

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UNIT
3

AIMS AND OBJECTIVES OF TEACHING SCHOOL MATHEMATICS

3.1 Introduction

Education is essential for achieving certain ends and goals. Learning of various subjects at school level are different means to achieve these goals. Mathematics has always held a key position in the school curriculum, because it has been considered knowledge indispensable to every human being. However, the knowledge of mathematics merely not meant for computational arithmetic and geometrical measurements but also played an important role in the education of all people. With the changing scenario of the World, mathematics has occupied an important role even in non-mathematical areas, such as social sciences, medical sciences, etc. With this new role, aims and objectives of teaching mathematics at school level have under gone tremendous changes from time to time according to the needs of scientific and technological oriented society.

The term 'aims of teaching mathematics,' stands for the goal or broad purpose that needs to be fulfilled by the teaching of mathematics in the general scheme of education. Aims are like ideals and their attainment needs long term planning. Therefore, they are divided into some definite functional and workable units named as objectives. The specific objectives are those short term, immediate goals and purposes that may be achieved within the specified classroom transactions.

In this Unit, we will discuss different aims and objectives of mathematics teaching at school level.

Learning Objectives

After studying this Unit, the student-teachers will be able to:

- explain the meaning and difference between goals, aims and objectives
- explain the need for establishing general aims and objectives of teaching mathematics
- state general aims and objectives of school mathematics education
- state general objectives of teaching mathematics at upper primary and secondary levels
- define taxonomies of educational objectives in general and to mathematics in particular
- write instructional objectives for teaching of mathematics at the
  - (i) Upper Primary Level
  - (ii) Secondary Level
- writing specific objectives and teaching points of various content areas in upper primary and secondary school mathematics .

3.2 Meaning and Difference between Goals, Aims and Objectives

Though in common usage goals, aims and objectives are used synonymously, they have in fact, different meanings.

Goals and Aims

The terms 'Goals' and 'Aims' are broad terms and are achievable over a long period of time. The difference between them is that goals are specific while aims are more general in nature.

The aim of education is all-round development of an individual, i.e., to develop an individual who can contribute to the society in all walks of life. But how can an individual contribute to the society? By becoming a teacher, a doctor, an engineer, a lawyer, a social worker or a mechanic etc. which may be termed as goals.

Aims are long term goals to be achieved by the students on completion of a course of study. Every subject included in the curriculum has distinct and unique aims intimately related to the broad aims or goals of education. To achieve these aims, certain means are required which may be called objectives. Thus, aims are the goals to be achieved and objectives are the means to achieve those aims.

Objectives

"I want to become a good teacher," is a goal, but to become a teacher, 'what all I should do to become a good teacher?' are the objectives to become a good teacher. To become a good teacher, I should have the subject competence, understanding of the learner, the learning process, and the competence in pedagogy.

Objective is a statement of actions through which an observable change is sought to be brought in the learner. Objectives also indicate the direction of pupil's growth and provide basis for selection of evaluation procedures. Objectives provide link between teachers, pupils and parents by focussing their attention with intended outcomes of learning. Thus, objectives determine the indicators to validate the process of education.

Specific objectives are subject or content based. What changes are expected to occur after the completion of a mathematics lesson(s)? When we write specific objectives in terms of behavioural terms, they are known as *behavioural objectives.* Behavioural objectives are measurable. They may be achieved in a certain period in the classroom. They are related to the expected change in the behaviour of the child. They are also known as *Instructional objectives* or *Teaching objectives* and are directly related to the teaching process. They are well defined, definite, clear, specific and measurable. They give direction to the learning process. Instructional objectives are concrete statements of the goals towards which instruction is directed. They play an important role in the process of learning and instruction.

Educational objectives are broad and philosophical in nature. They are related to school and educational system. According to B.S. Bloom (1956), "Educational objectives are not only the goals towards which the curriculum is shaped and towards which instruction is guided, but they are also the goals that provide the detailed specification for the curriculum and use of evaluation techniques."

The educational objectives that are achieved with the help of teaching are known as instructional objectives.

An objective is a part of the aim, thus it is narrower as compared to an aim. Aims and objectives both consist of two essential parts; an action verb and a subject content reference. They are written from the perspective of the learner, i.e., they are what the learner can do upon completion of learning. A simple example would be:

"On completion of the lesson, the learner should be able to solve the problems which involve calculation of area of a trapezium when the length of a pair of parallel sides and the distance between them is given".

Aims and objectives may be compared or differentiated on the basis of the following points:

- Aims are very broad and comprehensive, whereas objectives are narrower and specific.
- The main source of aims is philosophy or sociology, whereas for objectives psychology is one of the sources.
- Aims give direction to the educational system, whereas objectives give directions to the learner.
- Aims are the expectation of society and the nation whereas objectives are the expectations of the teachers.

- Aims are theoretical and indirectly influenced by teaching process while objectives are directly concerned with the teaching learning process.

3.3 Need for Establishing General Aims and Objectives of Teaching Mathematics

Aims and objectives are important components of the educational process, which assists in clarifying the relationship between the learner and the teacher. Without framing the objectives, the teacher may be more concerned with completion of the syllabus. But, by framing the objectives, the focus of the teachers can be shifted towards, 'How can the student grow and change through acquiring the knowledge, understanding and applying it in various life situations or to what level of learning the student can be taken.'

The writing of objectives assists teacher in selecting appropriate content, teaching strategies and assessment methods. These also help the students in knowing what is expected from them after completing the course.

3.4 General Aims of School Mathematics Education

Mathematics knowledge imparted should cultivate the values, such as development of concentration, the power of expression, attitude of discovery, self reliance, economical living and above all the quality of hard work as all these qualities are essential for a human being to survive in the world. Therefore, there is a definite place for mathematics in education. For all-round development of the child teaching of mathematics at school level should be very effective. For such effective and meaningful teaching of mathematics, it has to be dealt with constructive invention, motivation, intuition, construction of knowledge and its application through deduction and seeing the aesthetics in mathematics. The question is how to make teaching of mathematics effective and meaningful.

According to NCF-2005, the main goal of mathematics education in schools is the mathematisation of the child's thought processes. There are two aims of school mathematics—the narrow aim and higher aim.

The narrow aim of school mathematics is to develop 'useful' capabilities, particularly those relating to numeracy-numbers, number operations, measurements, decimal and percentage. The higher aim is to develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusions and to handle abstractions.

3.5 General Aims of Teaching of Mathematics – Classifications

Every teacher of mathematics needs to be informed and convinced about the educational aims of the subject. Her own conviction enables her to achieve the set goals or aims of teaching mathematics.

General aims of teaching of mathematics can be classified into three categories, i.e., utilitarian, disciplinary and cultural aims.

(a) Utilitarian Aims

Everybody irrespective of the fact to which class of society she/he belongs, makes use of mathematics in her/his daily life. Any person ignorant of mathematics can be easily cheated. So counting, addition, subtraction, multiplication, division, which are fundamental processes of mathematics have immense practical value. It has become one of the basic tools for business and commerce. A business woman uses the knowledge of percentage, average, stock shares to run her business efficiently. Thus, this aim makes the knowledge of mathematics functional and purposeful and establishes relationship between the subject and daily life.

- It makes the individual economical in using time, money, communication, etc.
- It develops a confidence, patience and self reliance.
- It nurtures the faculty of discovery and invention.

(b) Disciplinary Aims

Mathematical knowledge is exact, real and pure. It trains and disciplines the mind. It develops powers of reasoning and thinking, and reduces reliance on rote memory. Learning in mathematics possesses certain features which help the learner in developing characteristics of discipline like accuracy, simplicity, certainty of results, originality, reasoning, self evaluation and other by products of these characteristics like concentration, truthfulness, seriousness, etc.

(c) Cultural Aims

It is said, "Mathematics is the mirror of civilisation" It helps an individual to overcome difficulties in the way of her/his progress. The prosperity of an individual and her cultural advancement have depended considerably upon the advancement of mathematics. The modern civilisation owes its advancements to the progress of various occupations, such as agriculture, engineering, surveying, medicine, industry, navigation, road-rail building, etc. and contribution of mathematics in their advancement cannot be undermined. This aim helps the learner to understand the contribution of mathematics in the development of civilisation and culture. It has enabled him to understand the role of mathematics in fine arts and in beautifying human life. Learning of mathematics is a medium to pass on this heritage to the coming generations. Mathematics is also a pivot for cultural arts, such as music, sculpture, poetry and painting.

Mathematics shapes culture as a play back pioneer. Some of the important aspects of cultural heritage have been preserved in the form of mathematical knowledge.

Apart from these three types of major aims, it has some other types of fundamental aims as discussed below:

(1) **Vocational Aim:** The main aim of education is to help the learners to earn their living and to make themselves independent. This aim is called vocational aim. To achieve

this aim, the learning of mathematics is very important. Vocational aim helps to prepare learners for technical and other vocations where mathematics is applied. For example, engineering, architecture, accountancy, banking, business, agriculture, tailoring, carpentry, surveying and office work requires the knowledge of mathematics.

(2) **Social Aim:** Mathematics plays an important role in understanding the progress of society, and also to develop the society. The aims related to this process are called social aims.

(3) **Moral Aim:** Mathematics develops rational thinking and inturn this inculcates moral values among the human being.

3.6 General Objectives of Teaching Mathematics

3.6.1 General Objectives of Teaching Mathematics at Upper Primary Level

The students will be able to:

1. consolidate the mathematical knowledge and skills acquired at the primary level and acquire knowledge and understanding of the terms, symbols, concepts, principles, processes of mathematics
2. develop adequate skills of drawing, measuring, estimating, demonstrating and model making
3. develop abilities to consult and use mathematical, statistical tables and ready reckoners
4. acquire knowledge and understanding of practical geometry, simple mensuration and elementary algebra
5. develop abilities to read and interpret data from statistical graphs
6. apply mathematical knowledge and skills to solve common problems that occur in daily life and other simple mathematical problems
7. develop interest in mathematics
8. appreciate the contributions made by mathematicians in general and Indian mathematicians in particular.

3.6.2 General Objectives of Teaching Mathematics at the Secondary Level

Students will be able to:

1. consolidate the mathematical knowledge and skills acquired at the upper primary level
2. acquire knowledge and understanding of the terms, symbols, concepts, principles, processes and proofs
3. develop mastery of basic algebraic skills
4. develop drawing skills

5. develop ability to write/interpret algorithms for problem-solving
6. develop interest in mathematics and participate in mathematical club activities, quiz competitions, mathematics Olympiads at state and national levels
7. apply mathematical knowledge and skills to solve real life mathematical problems
8. apply mathematics by way of representing data in the form of graphs and charts, formulating and solving data based problems pertaining to actual data on population, agriculture, environment, industry, physical, biological and social sciences, engineering, defence, etc.
9. develop abilities to analyse, think and reason; and study mathematics systematically as a discipline
10. develop the ability to communicate the mathematical ideas precisely
11. develop necessary skills to use computers and mathematical software in learning and solving problems
12. develop reverence and respect towards great mathematicians for their contributions to the field of mathematics
13. develop appreciation for its brevity, preciseness, abstractness, structure and its deductive nature.

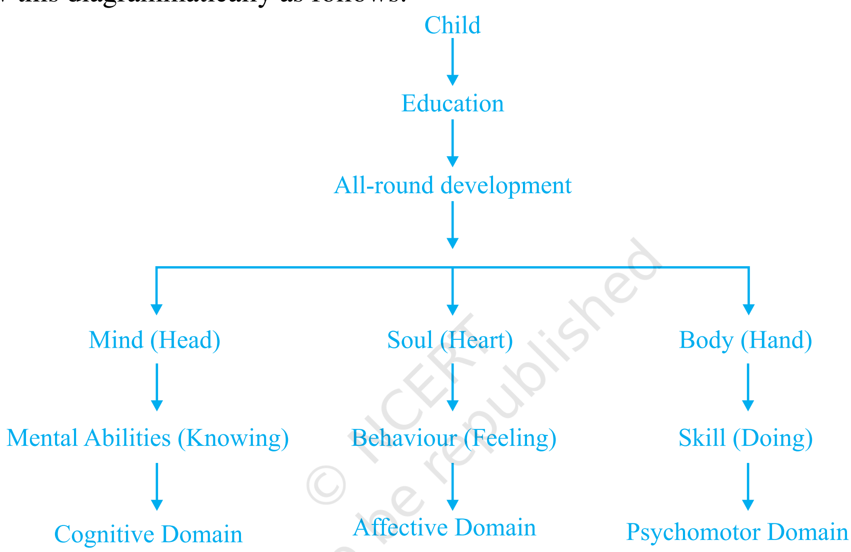
3.7 Taxonomies of Educational Objectives

The word 'taxonomy' has been derived from two Greek words '*taxis'* meaning 'arrangement' and 'nomas' meaning 'law'. Thus, taxonomy is a 'lawful' or an 'orderly arrangement.' The taxonomy of educational objectives are hierarchic in nature. Objectives are written in ascending order of complexity or difficulty. Each higher objective should subsume the lower ones. Besides, each higher level objective in sequence is in advance over the preceding ones. A few Taxonomies have been developed in the past. But the most acceptable and widely used are by Bloom (revised) and others for cognitive domain, by Krathwohl for affective domain and Dave for psychomotor domain.

RT (Revised Taxonomy) has taken into consideration the recent paradigm shifts in theories of learning which make learners more independent and responsible for their own learning (Constructivism, Meta-cognition and others). All these theories and approaches see learning as "a proactive activity, requiring self-initiated motivational and behavioural processes as well as meta-cognitive ones" (Zimmerman, 1998). *RT* has beautifully incorporated these new learner-centered learning paradigms into new framework with emphasis on learner's efforts to discover, construct and transform knowledge into a meaningful cognitive activity. Similarly, self-awareness, self monitoring and self-evaluation as in-built processes of meta-cognition are crucial levels in the hierarchy of *RT*. *RT* provides a framework of 'commutative hierarchy' of intellectual activities which brings measurable change in the learning behaviour of learners. It is generally applied to the cognitive domain of learning, which comprised the development of intellectual abilities and problem solving skills. Any pedagogical approach aims to recognize, implement and assess methodologies to move from presently achieved level of learning to more advanced level of learning and thus learner's readiness as mastery of pre-requisite concepts prepares the ground for next level of learning.

'Commulative hierarchy' of objectives in *RT* makes it pedagogically sound and empowers teachers to collaboratively engage in meaningful teaching-learning activities and reflective practices.

We know that education is the process of all-round development of an individual. We can show this diagrammatically as follows:



These entire three domains are interrelated as shown in Fig. 3.1.

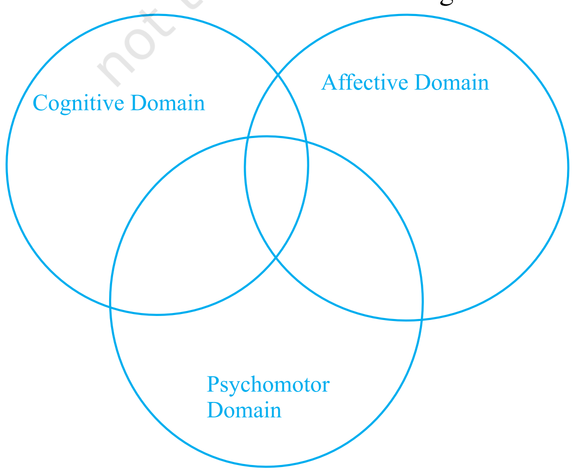


Fig. 3.1

This figure shows the interrelationship of these three domains. Now we will discuss them one by one in detail.

Cognitive Domain

The word cognitive here refers to *'knowledge'*. This domain consists of six classes of objectives arranged in a hierarchical order, in increasing levels of difficulty and sophistication and is cumulative in nature. Each level builds on and subsumes the ones below:

Remembering (recalling information) represents the lowest level in Bloom's taxonomy. It provides the basis for all higher cognitive objectives. Only after a learner is able to recall information, it is possible to move on to *understanding or comprehension* (giving meaning to information). The third level is *applying* which refers to using knowledge or principles in new or real life situations. The learner at this level, solves practical problems by applying information comprehended at the lower levels. The fourth level is *analysing*, i.e., breaking down complex information into simpler parts. The simpler parts, of course, were learnt at earlier levels of the taxonomy. *Evaluation* is the fifth level in Bloom's hierarchy. It consists of making judgments based on previous levels of learning to compare against a designated standard. The final level creating consists of *synthesising* something by integrating information that had been learnt at lower levels of the hierarchy.

Different categories of cognitive domain with their action verbs and specifications are given below:

1. **Remembering:** Remembering is the recall of specifics, processes, methods, patterns, structures, settings, generalisations, etc. The basic psychological process in use is remembering. This is distinguished from the remaining objectives which requires higher mental operations.

   Action Verbs: Select, measure, write, recall, recognise, state, define, list, narrate, recite, quote, etc.

Specifications

- Recall facts, definitions, laws, principles, procedures, etc.
- Recognises terms, laws, statements, processes, etc.

2. **Understanding:** Understanding is the next higher level objective to remembering. However, it is at the lower level from that of applying level. It enables to illustrate, compare, contrast and discriminate.

   Action Verbs: Select, present, describe, explain, illustrate, identify, substitute, distinguish, compare, classify, contrast, detect error, give example, translate, infer, etc.

Specifications

- Translates statement, passage, etc.

- Extrapolates given values, ideas, etc.
- Compares properties, functions, procedures, etc.
- Identifies as desired.
- Classifies as per given criteria.
- Distinguishes quantities, processes, etc.
- Explains the phenomena, etc.
- Calculates the answers.
- Plots given values.
- Arranges as per requirement.

3. **Applying:** Application pertains to the use of an abstract idea in a particular and concrete situation to arrive at a problem, generalisation of solution, etc. It includes both remembering and understanding.

   Action Verbs: Judge the sufficiency of the given data, select appropriate formula, predict, solve, apply, employ, verify, use, construct, assess, find and demonstrate.

Specifications

- Analyses unfamiliar situations.
- Applies knowledge and understanding to solve difficult problems.
- Generalises principles, methods, etc.

4. **Analysing:** Analysis connotes the breaking down of a complex situation into its constituent elements. Some people treat it as an equivalent of the objective of critical thinking which is only partly true. Analysis is an important component of critical thinking, but the latter is something more. Analysis may lead to the identification of elements, principles or relationships, etc.

   Action Verbs: Analyse, determine evidence, support, identify causes, divide, compare, criticise, discriminate and separate.

Specifications

- Analyses elements.
- Analyses relationship.
- Analyses process.
- Separates the complex ideas into its constituents.

5. **Evaluation:** Evaluation level objective calls for the operation of most complex mental processes necessary for judging a material, method or communication against the standard, which may be internal or external to it.

   Action Verbs: Apprise, justify, criticise, review, defend, assess, judge, avoid, identify, evaluate.

Specifications

- Judges in terms of internal criteria.
- Judges in terms of external criteria.
- Judges in terms of objective criteria.
- Judges in terms of subjective criteria.

6. **Creating:** Creating involves the ability to put together the elements or parts in such a way that a new pattern or theory emerges from it.

   Action Verb: Compile, generate, reconstruct, reorganise, produce, design, develop, synthesise, construct, create, build, summarise and select, organise and generalise.

Specifications

- Formulates a complex new idea out of ideas and concepts from multiple sources.
- Derives a set of abstract relationships.
- Develops new hypothesis, laws and principles.

The six levels of objectives mentioned above are arranged from simple to complex, easy to difficult. They have been further stated in terms of more specific behaviours. However, creating level though stated at level 6, need not be necessarily the highest level objective. It could be placed any where in the sequence since more often than not it is indifferent of other cognitive level objectives. The list of specifications can be extended further.

Affective Domain: The affective domain (Krathwohl, Bloom, Masia, 1973) includes the manner in which we deal with things emotionally, such as feelings, appreciation, enthusiasm, motivation, attitude, interest, emotion, values, mental tendencies and social adjustment of the students. The five major categories are listed below from the simplest behaviour to the most complex:

(1) **Receiving Phenomena:** Receiving is the lowest level objective of affective domain. At this level the student passively attends to particular phenomena or stimuli. She is aware that a phenomenon exists and she listens attentively. Specifications may be awareness, willingness to hear, selected attention.

Examples

- Listen to others with respect.
- Listen to and remember the names of newly introduced people.

  Key Words: Chooses, describes, follows, holds, identifies, locates, points to, sits and erects.

(2) **Responding to Phenomena:** Responding is the next level objective. It expects greater motivation, regularity in attention and active participation on the part of the learners. They attend and react to a particular phenomenon. It may also be described

as interest to respond to a particular statement or phenomenon. Learning outcomes may emphasise compliance in responding, willingness to respond or satisfaction in responding (motivation).

Examples

- Participates in classroom discussions.
- Gives a presentation.
- Questions new ideals, concepts, models, etc. in order to fully understand them.
- Practices the safety rules.

Key Words: Answers, assists, aids, complies, conforms, discusses, greets, helps, labels, performs, practices, presents, reads, recites, reports, selects, tells and writes.

(3) **Valuing:** Valuing is the third level objective which involves increasing internalisation of the worth or value a person attaches to a particular object, phenomenon or behaviour. This ranges from simple acceptance to the more complex state of commitment. Valuing is based on the internalisation of a set of specified values, while clues to these values are expressed in the learner's overt behaviour and are often identifiable.

Examples

- Demonstrates belief in the democratic process.
- Is sensitive towards individual and cultural differences (value diversity).
- Shows the ability to solve problems.
- Proposes a plan to social improvement and follows through with commitment.
- Informs management on matters that one feels strongly about.

Key Words: Completes, demonstrates, differentiates, explains, follows, forms, initiates, invites, joins, justifies, proposes, reads, reports, selects, shares, studies and works.

(4) **Organisation:** Organisation is the fourth level objective which brings together different values, resolving conflicts between them, starting to build an internally consistent and a unique value system or attitude. An individual's behaviour is not ordinarily motivated by a single attitude. Development of ones' own code of conduct or standard of public life may be an instance of the organisation of a value system. The emphasis is on comparing, relating and synthesising values.

Examples

- Recognises the need for balance between freedom and responsible behaviour.
- Accepts responsibility for one's behaviour.
- Explains the role of systematic planning in solving problems.
- Accepts professional ethical standards.

- Creates a life plan in harmony with abilities, interests and beliefs.
- Prioritises time effectively to meet the needs of the organisation, family and self.

  Key Words: Adheres, alters, arranges, combines, compares, completes, defends, explains, formulates, generalises, identifies, integrates, modifies, orders, organises, prepares, relates and synthesises**.**

(5) **Internalising Values (Characterisation):** Characterisation is the highest level objective. This is the highest ability achievable objective in the affective domain. This is attained when an individual consistently found behaving in accordance with the values or attitudes she has imbibed. She has a value system that controls their behaviour. The behaviour is pervasive, consistent, predictable and most importantly characteristic of the learner. Instructional objectives are concerned with the student's general patterns of adjustment (personal, social and emotional).

Examples

- Shows self reliance when working independently.
- Cooperates in group activities (displays teamwork).
- Uses an objective approach in problem solving.
- Displays a professional commitment to ethical practice on a daily basis.
- Revises judgments and changes behaviour in light of new evidence.
- Values people for what they are, not how they look.

  Key Words: Acts, discriminates, displays, influences, listens, modifies, performs, practices, proposes, qualifies, questions, revises, serves, solves and verifies.

  Thus, the affective domain includes those objectives which are concerned with the development of attitude, interests, values, appreciations, adjustment, etc.

Psychomotor Domain: The psychomotor domain (Dave 1975) includes physical movement, coordination and use of the motor skill areas. This is the third part of the taxonomy and includes the manipulative and motor skill areas. The psychomotor domain concerns itself with levels of attainment on neuro-muscular coordination. As the level of coordination goes up, the action becomes more refined, speedy and automatic. The physical actions involve in handwriting, playing, using equipments, making outline, drawing figures and many others. Development of these skills requires practice and is measured in terms of speed, precision, distance, procedures or techniques in execution. The five major categories are listed from the simplest behaviour to the most complex:

(1) **Imitation:** Imitation is the lowest level objective of psychomotor domain. It starts as an inner push or impulse. It is represented by covert inner rehearsal of the muscular system which may be taken to be more of an action at the mental level. Soon it may grow into an overt act with capacity to repeat the performance with very rudimentary coordination. It involves observing and patterning behaviour after someone else. Here performance may be of low quality.

Examples

- Initiation
- Repetition
- Copying a work/task

(2) Manipulation: Manipulation is the next higher level objective of psychomotor domain. It involves follow up of directions, selecting certain actions in preference to others and acting accordingly. It enables an individual to perform certain actions by following instructions and practising. It marks the beginning of the fixation of operation at the end of initial fumbling in the manipulative actions.

Examples

- Follow directions
- Manipulation
- Selection
- Fixation
- Creating work on one's own, after taking lessons or reading about it.

(3) Precision: Precision is the third level objective and is to be attained when reproduction of operations is carried out with speed and refinement giving the learner the ability to control (increase, decrease or modify) her action according to requirement. Working and reworking something, so it will be 'just right.'

Examples

- Reproduction
- Control
- Refinement.

(4) Articulation: Articulation is the fourth level objective, which can be said to be attained when the learner is able to handle a number or series of actions, keeping in view their sequence and rhythm. It involves coordination in action, i.e., right sequence in right proportion of time or at the right moment. It coordinates a series of actions, achieving harmony and internal consistency.

Examples

- Sequencing of actions
- Coordination of actions
- Completion of task

(5) **Naturalisation:** Naturalisation is the final stage which is the equivalent of a perfect habituation ranging from automatisation to routinisation. At this level, the performer does not remember the order of operations. Her actions are more or less mechanical and without any conscious thinking or planning. She has high level of performance which becomes natural without needing to think much about it. It is a mere reflex action.

Examples

- Automatisation
- Internalisation

Thus, in psychomotor domain the focus is on development of motor skill. The physical actions involved in handwriting, playing, using equipments, making outline, drawing figures/graphs are in the psychomotor domain.

3.8 Instructional Objectives at Different Levels

Though Bloom's Taxonomy of educational objectives in the cognitive domain has six levels of remembering, understanding, applying, analysing, evaluating and creating, as adapted to mathematics, it has only three, levels, namely remembering, understanding and applying. Here routine exercises and applications are included under knowledge and understanding depending on the complexity and applying level objectives, include non-routine applications, analysis, creating and evaluation.

In view of the above, for writing instructional objectives we have employed following levels in the cognitive, affective and psychomotor domains: (1) Remembering, (2) Understanding, (3) Applying, (4) Skills, (5) Appreciation, (6) Interest, and (7) Attitudes.

3.8.1 Upper Primary Stage

1. **Remembering Objectives:** To remember terms, facts, concepts, symbols, definitions, principles and formulae of mathematics and also applications which are routine in nature requiring probably only substitution in a formula.

   Specifications: To demonstrate the achievement of above objectives, the pupil:

1.1 Recalls the terms, facts, definitions, formulae, concepts, processes, etc.

1.2 Recognises formulae, figures, concepts, procedures and processes, etc.

1.3 Solves routine type of problems using a formula.

2. **Understanding Objectives:** To develop understanding of concept, principles and processes of mathematics, to apply principles and processes in routine situations.

Specifications: To demonstrate the achievements of the above objectives, the pupil:

2.1 gives illustrations, detects errors in a statement, formulae or figure and corrects them

2.2 explains concepts, principles and configurations

2.3 discriminates between closely related concepts and principles

2.4 classifies as per given criteria

2.5 identifies relationship among the given data

2.6 translates variable statements into symbolic relationship and vice–versa

2.7 estimates the results

2.8 interprets given charts, graphs and data

2.9 verifies properties

2.10 solves routine type of problems using concepts and processes.

3. Applying Objectives: To apply the acquired knowledge and understanding of mathematics in unfamiliar situations or new problems.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

3.1 analyses and solves non-routine type of problems

3.2 finds out the adequacy or relevance of data

3.3 establishes relationship among the given data

3.4 selects the appropriate method for solving a problem

3.5 suggests alternative method of solution

3.6 gives justification to the method of solution

4. Skill Objectives: To acquire different skills (a) computing (b) drawing geometrical figures and graphs (c) reading tables, charts, graphs, etc.

Specifications: To demonstrate the achievements of the above objectives, the pupil:

4.1 carries out oral calculations with ease and speed

4.2 does written calculations with ease and speed

4.3 handles geometrical instruments with ease and proficiency

4.4 measures accurately

4.5 draws free hand diagrams with ease

4.6 draws figures accurately

4.7 draws figures according to scale

5. Appreciation Objectives: To appreciate the use of mathematics in day-to-day life and in other disciplines.

Specifications: To demonstrate the achievement of the above objectives, the pupil :

5.1 appreciates the use of mathematics in other disciplines

5.2 appreciates the symmetry of figures and patterns

5.3 appreciates qualities like brevity and exactness through the study of mathematics

6. Interest Objectives: To develop interest in mathematics.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

6.1 solves mathematical puzzles

6.2 participates in the activities of mathematics club

6.3 reads additional material in mathematics

6.4 formulates additional problems.

7. Attitude Objectives: To develop scientific attitude through the study of mathematics.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

7.1 examines all the aspects of a problem

7.2 points out errors boldly if convinced

7.3 accepts errors without hesitation

7.4 respects the opinions of others

3.8.2 Secondary Stage

1. Remembering Objectives: To acquire the knowledge of terms, facts, concepts, symbols, definitions, principles and formulae of mathematics.

Specifications: To demonstrate the achievements of the above objectives, the pupil:

1.1 recalls, terms, facts, definitions, formulae, concepts, processes

1.2 recognises formulae, figures, concepts, procedures and processes

2. Understanding Objectives: To develop the understanding of the concepts, principles and processes of mathematics.

Specifications: To demonstrate the achievements of the above objectives, the pupil:

2.1 gives illustration

2.2 detects errors in a statement, formula or figure and corrects them

2.3 explains concepts, principles and configurations

2.4 discriminates between closely related concepts and principles

2.5 classifies as per given criteria

2.6 identifies relationship among the given data

2.7 translates variable statement into symbolic relationship and vice-versa

2.8 estimates the results

2.9 reads and interprets the given charts, graphs and data

2.10 verifies properties, indicates hypothesis

2.11 solves routine type of problems using concepts, principles, etc.

2.12 reads and interprets data from a table

2.13 uses calculator and computer

3. Applying Objectives: To apply the acquired knowledge and understanding of mathematics in unfamiliar situations or new problems.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

3.1 analyses and solves problems

3.2 finds out the adequacy or relevance of data

3.3 establishes relationship among the given data

3.4 selects the appropriate method for solving the problem

3.5 suggests alternative method of solution/proof

3.6 gives justification for the method of solution

4. Skill Objectives: To acquire different skills like, (a) computing, (b) drawing geometrical figures and graphs, (c) reading tables, charts, graphs, etc.

Specifications: To demonstrate the achievements of the above objectives, the pupil:

4.1 does written calculations with ease and speed

4.2 handles geometrical instruments with ease and proficiency

4.3 measures accurately

4.4 draws free hand diagrams with ease

4.5 draws figures accurately and according to scale

4.6 reads data/table with speed and accuracy

5. Appreciation Objectives: To appreciate the use of mathematics in day-to-day life and other disciplines.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

5.1 appreciates the use of mathematics in other curricular areas

5.2 appreciates the symmetry of figures and patterns

5.3 appreciates qualities like brevity and exactness

5.4 appreciates the contributions of mathematicians in general and Indian mathematicians in particular.

6. Interest Objectives: To develop interest in mathematics.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

6.1 solves mathematical puzzles

6.2 participates in the activities of mathematics club

6.3 reads additional material in mathematics

6.4 constructs additional problems

7. Attitude Objectives: To develop scientific attitude through the study of mathematics.

Specifications: To demonstrate the achievement of the above objectives, the pupil:

7.1 accepts a proposition only when logically proved

7.2 examines all the aspects of a problem

7.3 points out errors

7.4 accepts errors without hesitation

7.5 respects the opinion of others

7.6 keeps an open mind and develops the habit of logical thinking

3.9 Writing Learning (Specific) Objectives and Teaching Points of Various Content Areas in Mathematics

Writing specific objectives is the initial stage of planning for teaching and learning. When you have identified appropriate objectives, the next task is to consider teaching strategies relevant to the nature of the learnings expected and to choose assessment methods that reflect the action verbs you have used. Objectives should be reviewed regularly in the light of personal experiences with teaching and information collected from students, either informally or via written evaluations.

Whenever teaching is done in the classroom for the achievement of certain objectives, the teacher should select a problem situation suitable for the situation of the subject matter with which desirable changes in learning capacity of the students could take place. Here 'changes in learning capacity means observable changes exhibited by the student as a result of the teaching learning process. The learning objectives of the content, popularly known as specific objectives are systematic planning of experiences which enable learners to meaningfully construct knowledge for themselves.

After formulation of objectives, the next step is to express them in specific terms in relation to what the learners are expected to do. The teacher should state to 'what changes' and 'what aspects' are to be brought in the learner in order to achieve the objectives in clear and simple terms. Let us take some examples.

Example 1

Topic: Solution of a linear equation by graphical method

1. Remembering Objectives

(a) The pupil will be able to recognise linear equations.

(b) The pupil will be able to understand the meaning of a linear equation.

2. Understanding Objectives

(a) The pupil will be able to give examples of linear equations.

(b) The pupil will be able to understand the geometrical construction.

3. Applying Objectives

(a) The pupil will be able to represent linear equation graphically.

(b) The pupil will be able to solve a linear equation through graphical method.

4. Skill Based Objectives

- The pupil will be able to draw the graph of a linear equation choosing a proper scale

Example 2: Suppose the teacher has to teach 'permutations'. Then the objectives are as follows:

1. Remembering Objectives

- The pupil will be able to recall the formula for calculating the number of arrangements of objects.

2. Understanding Objectives

(a) The pupil will be able to cite examples of permutations

(b) The pupil will be able to define permutations.

(c) The pupil will be able to explain the formula of permutations.

3. Application Objectives

(a) The pupil will be able to apply the knowledge of permutations in solving daily life problems.

(b) The pupil will be able to solve the problems involving permutations.

Example 3

Topic: Volume and surface area of a cylinder, cone, sphere and hemisphere.

1. Remembering Objectives

The pupil will be able to:

1. Recall the formulae for calculating curved surface area, total surface area and volume of a cylinder

2. Recall the formulae for calculating curved surface area, total surface area and volume of a cone
3. Define slant height of a cone
4. State the formulae for total surface area and volume of a sphere as well as of a hemisphere

2. Understanding Objectives

The pupil will be able to:

1. Gives examples of the objects from our daily life which suggests the concept of right circular cylinder, cone, sphere and hemisphere
2. Distinguish between the features of a cylinder and a cone
3. Find the curved surface area and total surface area of the cylinder and cone
4. Find out the total surface area and volume of a sphere as well as of a hemisphere.

3. Applying Objectives

The pupil will be able to:

1. Compute the surface area and volume of a cylinder that is made by folding a rectangular sheet of given length and breadth (by taking height as (i) length (ii) breadth)
2. Calculate the cost of canvas required to make a conical tent, when related information is given
3. Find the cost of plastering or painting a cylindrical pillar, when the related information is given
4. Find radius or height of the cone when a given cylinder is melted into cone or vice-versa
5. Compute the number of lead shots or spherical balls formed from a solid conical object when radius of lead shots is given.

4. Skill Based Objectives

The pupil will be able to:

1. Draw the figure of a cylinder.
2. Draw the figure of a cone.
3. Draw the figure of a sphere.

Example 4

Topic: Exponents and Radicals

Specific Objectives of Unit

The following objectives will be achieved after teaching the entire unit of exponents and radicals.

Remembering Objectives

The pupil will be able to:

- list laws of exponents (verbal/symbolic)
- state the definition of radical and radicands
- state the definition of mixed and pure radicals
- represent the general form of exponents in radical forms.

Understanding Objectives

The pupil will be able to:

- differentiate between mixed and pure radicals
- classify given data as pure and mixed radicals
- give examples of radicals and radicands
- identify the radicals and radicands in the given expression
- translate verbal statements of various laws into symbolic representations

Applying Objectives

The pupil will be able to:

- simplify problems by selecting the appropriate laws or various combinations of laws
- simplify expressions involving rationalisation.

Skill Based Objectives

The pupil will be able to:

- carry out oral calculations using simple exponential laws while solving complex problems with ease and speed
- carry out written calculations with ease, speed and efficiency
- compute radical free expressions by using rationalisation.

Example 5

Topic: Probability

Specific Objectives of Unit

The following objectives will be achieved after teaching this Unit on probability.

Remembering Objectives

The pupil will be able to:

- recall the definition of probability.

- state the formula for probability.
- state the condition for probability of an event to occur and not to occur.

Understanding Objectives

The pupil will be able to:

- translate verbal statement into symbolic representation.
- cite examples from daily life involving probability.
- identify objects like dice, playing cards etc., where probability of outcomes is to be calculated.
- discriminate between probability of an event to occur and not to occur.

Applying Objectives

The pupil will be able to:

- solve problems related to probability.
- establish relationship between probabilities of an event to occur or not to occur.
- select appropriate method to solve problems related to probability.

Skill Based Objectives

The pupil will be able to:

- carry out oral calculations with ease and speed like – probability of occurring an event.
- carry out written calculations easily by analysing various events.
- perform experiments related to probability in mathematics laboratory.

In constructivist paradigm, the process of writing objectives is slightly different. Let us take some examples.

Example 1

Topic: Perimeter

Objectives

- To reinforce the idea of perimeter as boundary of the figure among learners by using Geoboard.
- To enable learners to calculate perimeter of closed figures (using Geoboard).
- To enable the learner to develop the formulae for calculating perimeters of a rectangle, square and triangle.
- To enable learners to use the formulae to calculate/solve problems related to perimeter.

Example 2

Topic: Circumference of a circle.

Objectives

- To enable learners to understand the concept of a circle.
- To enable learners to find the relationship between the circumference and the diameter of a circle.
- To enable learners to solve problems related to circumference of circle using its formula.
- To enable learner to differentiate between area and perimeter of a circle.

Example 3

Topic: Area of a rectangle

Objectives

- To enable learners to develop the formula for area of a rectangle and hence, of a square.
- To enable learners to solve simple problems related to area of a square and a rectangle.

Example 4

Topic: Area of a parallelogram

Objectives

- To enable learners to develop the formula for the area of parallelogram.
- To enable learners to solve problems using formula for area of parallelogram.

Example 5

Topic: Area of a triangle

Objectives

- To enable learners to develop the formula for area of a triangle.
- To enable learners to solve various problems using the formula for area of triangle.

Example 6

Topic: Algebraic expressions

Objectives

- To enable learners to identify a variable in an algebraic expression.
- To enable students to state the meaning of term 'variable' and 'constant'
- To enable learners to define an algebraic expression.
- To enable learners to state the components of an algebraic expression.
- To enable students to translate a given statement into an algebraic expression.
- To enable learners to identify terms of an algebraic expression.

- To enable learners to differentiate between like and unlike terms.
- To enable learners to state the meaning of coefficient of a term and hence, enable them to write coefficients of the terms in an algebraic expression.
- To enable learners to classify algebraic expressions into monomial, binomial, trinomial etc.

Teaching Points

Meaning of a constant and variable, meaning of an algebraic expression, term of an algebraic expression, like and unlike terms, coefficient of a term, classification of algebraic expression as monomial, binomial, etc.

You can now write the teaching points for other topics in mathematics.

EXERCISE 3.1

1. Differentiate between aims and objectives of teaching mathematics.
2. What are the aims of teaching of mathematics at different levels of school education? Explain, to what extent these aims are being achieved in our schools.
3. What should be the general aims of teaching of mathematics at school level? Which efforts would you like to make to achieve them, being a teacher?
4. Discuss briefly the educational value of teaching mathematics at secondary level.
5. Clarify with examples the educational objectives of teaching mathematics at secondary level.
6. Explain in short the general objectives of teaching mathematics at different levels of school education.
7. What are the aims and objectives of teaching mathematics at secondary stage? Write down five objectives in terms of specific learning outcomes each for remembering, understanding, applying and skill, on the topic ‘Graph of a Linear Equation in two variables’.
8. Explain the advantages of writing instructional objectives in behavioural terms. Illustrate with examples, the role of action verbs in writing behavioural objectives. Write down five behavioural objectives each for remembering, understanding and applying of selected concepts in mathematics.
9. Discuss the relationship between instructional objectives and evaluation. Elaborate your answer with the help of suitable examples.
10. What are the objectives of teaching mathematics at different school levels? Out of these, explain any two in detail.

11. Write the specific objectives for the following units:
    (i) Ratio and proportion
    (ii) Exponents and radicals
    (iii) Volume and surface area of a cylinder
    (iv) Probability

Summary

In this Unit, we discussed the aims and objectives of teaching mathematics. Aims of teaching mathematics are classified into utilitarian, disciplinary, cultural, vocational, social and moral aims.

Objectives of teaching Mathematics have been divided into six levels, namely

- remembering objectives
- understanding objectives
- applying objectives
- skill based objectives
- attitude objectives, and
- appreciation objectives.

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UNIT

4

SCHOOL MATHEMATICS CURRICULUM

4.1 Introduction

The word 'curriculum' is said to be derived from a Latin word '*currere*', which means 'a race course to run'. So, curriculum is considered to be an educational programme (or a course of study) to be followed for reaching a certain goal or destination. This course of study is not done in vacuum. It is done by providing planned activities and experiences to the students, as per needs of the society, to ensure optimum human resource development of a particular country. Thus, it can be said that curriculum is a sequence of planned activities and experiences made available to the students based on certain predetermined objectives. It is also worth noticing that needs of the society are changing from time to time and, therefore, curriculum development is a continuous and dynamic process. In this Unit, we shall discuss different objectives of curriculum, principles of curriculum designing and other aspects of curriculum at different stages of schooling with special reference to mathematics and its branches like arithmetic, algebra, geometry, etc.

Learning Objectives

After studying this Unit, student-teachers will be able to:

- understand the meaning of curriculum
- differentiate between curriculum and syllabus
- state different objectives of mathematics curriculum at various stages of schooling

- state various principles and methods of designing mathematics curriculum at various stages of schooling
- state and explain the highlights and concerns pertaining to mathematics curriculum at the school stage
- select and analyse various topics for the mathematics syllabus of different stages of schooling
- construct syllabi of mathematics at different stages of schooling
- perform pedagogical analysis of different topics in mathematics for developing textbooks at different stages of schooling.

4.2 Syllabus, Curriculum and Curriculum Framework

Generally, 'curriculum' is considered to be synonymous with 'syllabus'. In fact, these two concepts are quite different. Syllabus refers to the content of each subject in the curriculum, while a curriculum outlines what is to be taught and the knowledge, skills and attitudes which are to be deliberately fostered, together with stagewise objectives. Curriculum has been defined by different authors in different ways. Some of these are as follows:

- Curriculum is a collection of all those aspects of schooling which are deliberately planned
- Curriculum is a judiciously organised content related to different subjects
- Curriculum is an organised collection of processes, procedures, programmes and the other similar activities which are applied to pupils to acquire certain types of predetermined objectives
- Curriculum is a teaching strategy, which in turn, is conceived of as being a series of goal oriented activities or procedures to be carried out by the teachers in a class of students in the context of a syllabus or a body of subject matter
- Curriculum is the sum total of all educational experiences that students have in a school
- Curriculum is a tool in the hands of an artist (teacher) to mould his material (students) in accordance with his/her ideals in his/her workplace (the school)

As per an educational dictionary, curriculum is defined as the total structure of ideas and activities developed by an educational institution to meet the needs of the students and to achieve educational aims. So, we may say that curriculum not only covers the syllabus, i.e., a broad outline of courses of study but also covers wider areas of individual and group life relating to the various educational and content areas. Thus, a curriculum includes content, pedagogy, systemic characterstics and assessment.

A curriculum is developed on the guidelines provided in the form of a policy document developed by educational experts as per needs of the society. This document is usually called a 'Curriculum Framework'. It interprets educational aims vis-a-vis both individual and society to arrive at an understanding of the kind of learning experiences that school teacher should provide to the students.

EXERCISE 4.1

1. What do you understand by a Curriculum?
2. What is a Curriculum Framework?
3. What is the difference between a Curriculum and a Curriculum Framework?
4. What is the difference between a Curriculum and a Syllabus? Explain it through an example.

4.3 Mathematics Curriculum and its Objectives

Mathematics curriculum forms an important part of the overall school curriculum. It has always occupied an important place at the school stage due to its decisive role in building up our civilisation. Mathematics is a self contained independent discipline with its own language and structure. Besides being an independent discipline, it has a lot of applications in other branches of knowledge.

During the last four decades, there has been an explosion of knowledge all over the World with tremendous development not only in the field of science, technology and electronics, but also in commerce, industries, factories, railways, posts and communications. In all these developments, an increasing applications of mathematics is clearly evident.

Due to above reasons, The Education Commission (1964-66) emphasised the importance of mathematics education from primary school stage to research degree levels and suggested various guidelines for future mathematical activities in schools, colleges and universities. The Commission was of the view, that if we want to survive as a Nation in the present complex technological World, the proper foundation of mathematics must be laid in the schools. The Commission also observed that in the teaching of mathematics, more emphasis should be on the understanding of basic principles rather than on the mechanical teaching of mathematical computations. It also recommended that the study of mathematics should be compulsory for all the students up to the secondary school stage.

The National Policy on Education (NPE) 1986 also proposed making mathematics as a compulsory subject of study upto the secondary stage for all children. According to *National*

Curriculum Framework (NCF) 2000, one of the basic aims of teaching mathematics in schools is to inculcate the skill of quantification of experiences around the learners. Mathematics helps in the process of decision-making through its applications to familiar as well as non-familiar real-life situations. It contributes in the development of precision, rational and analytical thinking, reasoning, positive attitude and aesthetic sense. Apart from being a distinct area of learning, it helps enormously in the development of other disciplines also, which involve analysis, reasoning and quantification of ideas.

As per guidelines of *National Curriculum Framework (NCF)* 2005, the main goal of mathematics education is to build abilities of mathematisation in students. It is worth mentioning that the rapid spread of modernisation in science, technology and even in social sciences is due to the process of mathematisation. NCF – 2005 has also suggested two types of aims for school mathematics education:

(i) **Narrow Aim:** Development of basic skills which include development of useful capabilities, particularly related to numeracy, numbers, number operations, measurement, decimals and percentages.

(ii) **Higher Aim:** Development of mathematical thinking which includes development of the child's resources to think and reason mathematically, to prove assumptions to logical conclusions and to handle abstractions.

This calls for a curriculum which is ambitious, coherent and which teaches important mathematics. It should be *ambitious* in the sense that it seeks to achieve the higher aim mentioned above rather than achieving only narrow aim. It should be *coherent* in the sense that the variety of methods and skills available piecemeal (in arithmetic, algebra etc.) cohere into an ability to address the problems that come from other domains, such as sciences and social sciences. It should be *important* in the sense that the students feel the need of such problems that teachers and students find it worth their time and energy to address these problems.

In view of the above discussions, we may have the following main objectives of mathematics curriculum at the school stage:

The students

- attain proficiency in fundamental mathematical skills
- comprehend basic mathematical concepts
- develop desirable attitudes to think, reason, analyse and articulate logically
- acquire efficiency in sound mathematical applications within mathematics and in other subject areas

- attain confidence in making intelligent and independent interpretations
- appreciate the power and beauty of mathematics for its application in sciences, social sciences, humanities and arts.

EXERCISE 4.2

1. According to NPE – 1986, upto what level mathematics should be compulsory in Indian schools? Give your opinion on it.
2. What is the basic aim of mathematics teaching in schools as suggested in NCF – 2000?
3. What is the main goal of mathematics education mentioned in NCF – 2005?
4. What are the two aims of mathematics education as per guidelines of NCF – 2005? Explain these in the light of the nature of mathematics.
5. In what sense should mathematics curriculum be

   (i) ambitious? (ii) coherent? (iii) important?
6. List main objectives of mathematics curriculum at the school stage. Do you agree with these objectives? Give reasons in support of your answer.

4.4 Principles of Designing Mathematics Curriculum

As already stated, curriculum is a sequence of planned activities and experiences available to the students based on certain predetermined objectives or goals. Designing (or developing) a curriculum involves answering the following questions:

(i) What are educational goals?

(ii) What are the educational experiences to be provided to achieve these goals?

(iii) How can these educational experiences be organised in an effective manner?

(iv) How far have the set goals been achieved through these experiences?

Thus, for designing a curriculum, there is a need of taking decisions regarding the following:

(i) Syllabus content, (ii) Classroom experiences or pedagogical style and (iii) Evaluation techniques.

The above three decisions are then knit together into experiences which provide a student the structures necessary to make classroom experiences effective.

The curriculum designing involves the following two stages:

(I) Curriculum Construction

(II) Curriculum Organisation

(I) Curriculum Construction

By curriculum construction, we mean the selection of appropriate topics to be included in the curriculum. There are certain basic principles which should form the basis for this selection, i.e., for construction of a good mathematics curriculum. They are as given below:

(a) Curriculum must be Child Centered

For this, curriculum should be based on present needs and capabilities of the children. It must provide an environment conducive to learning where children feel secure, where there is absence of fear and which is governed by relationship of equality and equity. The curriculum must enable children to find their voices, nurture their curiosity to do things, to ask questions and to pursue investigations, sharing and integrating their experiences with school knowledge. In other words, the curriculum must be helpful in developing an initiative, cooperation and social responsibilities among the children. That is, the curriculum must meet the physical, intellectual, emotional and social needs of the children.

(b) Curriculum must be Dynamic

It means that curriculum must include the latest developments in mathematics and in other areas, such as sciences, social sciences and information technology.

The data used in the instructional material must be upto date. For example, it should not include outdated data like "price of the sugar is Rs. 1.50 per kg. or "salary of a person is Rs.20 per month" and so on. Similarly, phrases like "Angle subtended by an arc at the circumference" should not be used. Instead, it should be "Angle subtended by an arc at any point on the circle". Further, it should be as per requirement of present as well as future age of globalisation and information technology. For this purpose, problems based on speed and distance, statistics, probability requiring the use of ICT should also find a place in the curriculum.

On the whole, it should be free from temporal and spatial variables/aspects.

(c) Curriculum must be Related to Everyday Life

It means that the curriculum must provide sufficient opportunities to the children for relating their classroom learning to their daily life experiences. It must emphasise that there is no science, no art and no profession where mathematics does not hold a key position.

(d) Curriculum must be Rationalistic

It means that the curriculum must inculcate rational and original thinking in the minds of the children. It is a well known fact that mathematics is a science of logical reasoning which possesses a number of characteristics, such as simplicity, accuracy, originality, etc.

(e) Curriculum must Lay Emphasis on 'Learning for Living' Instead of 'Living for Learning'

It means that learning for the sake of learning should not be encouraged. In fact, it should be need based. In view of the above, much stress should not be laid on topics such as nine-point circle, divisibility rules by larger prime numbers, complicated problems on time and work, time and distance, etc. at the school stage.

(f) Curriculum must Help in Preserving and Transmitting Our Cultural Heritage

It means that curriculum must include some activities which help in preserving the cultural heritage of the country. The culture is to be understood in the true sense. Scientific and social principles are associated with culture and their understanding requires knowledge of mathematics.

(g) Curriculum must be Flexible

It means that curriculum must include activities for the requirements of different types of children (such as children of rural and urban areas, of different socio-economic strata). It ensures that the marginalised groups of students are able to relate the curriculum with their experiences and native wisdom so that they can overcome their disadvantages and be able to perform at par with everyone.

(h) Curriculum must be Well Integrated

It means that there should be continuity and coherence within mathematics as well as with other subjects also.

Keeping in view the above principles, mathematics curriculum is then constructed by selecting appropriate topics.

(II) Curriculum Organisation

Curriculum organisation means arranging the topics selected among different Classes and within a Class.

Thus, after appropriate selection of topics, the curriculum is organised, maintaining a proper mathematical sequence and continuity among them. This organisation is done keeping in view the following principles:

(a) Principle of Correlation

While organising a mathematics curriculum, following types of correlation must be kept in mind:

- correlation with daily life
- correlation with other subjects
- correlation among different branches of mathematics
- correlation among different topics of the same branch of mathematics.

For example, topics like ratio, proportion, percentage, simple and compound interest, discount, etc. are essential for daily life. Vectors, differentiation and integration are essential for the study of Physics; Similarity of triangles is essential for the study of trigonometry and so on.

(b) Principle of Motivation

Organisation of the curriculum must motivate the students to learn. For this purpose, content presented should be interesting, exciting and challenging. Games and puzzles should also be included for this purpose. Famous anecdotes and contributions of famous mathematicians, particularly Indian mathematicians should be included at appropriate stages in the curriculum.

(c) Principle of the Degree of Difficulty

Content should be presented in the increasing order of difficulty. The difficulty level of the topics must be decided on the basis of the capabilities of the students for a particular level. In addition to the above, the questions in a particular exercise of a textbook should also be presented in an increasing order of difficulty after proper grading.

(d) Principle of Logical and Psychological Orders

Organisation of the mathematics curriculum should be done by keeping an appropriate balance between the logical and psychological orders of the topics. It means that there should be a sequential development of the topics which is most appropriate for the students of that particular age level.

Logical order involves sequenced and systematic organisation of the content, while psychological order is dictated by level of intuition, need, curiosity and the interest of the students. Logical order is a feature of the traditional type of curriculum, while psychological order is a characteristic of new approach to curriculum. For example, in the present curriculum, theorems in geometry are taught only after these have been sufficiently explained through experiments and activities. It is also suggested that study of geometry should start from solids to surfaces and lines and later to abstract concepts.

For similar reasoning, it is suggested that reading and interpretation of graphs should precede their constructions. However, in certain cases, logical and psychological orders go together.

(e) Principle of Vertical Correlation

By vertical correlation, we mean that the organisation of a content for a particular Class must be in continuation with the topics covered in the previous Classes and also it should form a basis for the organisation of the content for the higher Classes. Further, topics included in a particular Class may be arranged in order from simple to complex.

(f) Principle of Activity

It is a fact that “learning by doing” makes learning more meaningful and permanent.
Therefore, a number of activities should also be included while organising a mathematics curriculum. These activities will help in relating abstract mathematical concepts to concrete

objects and will also induce enthusiasm and interest among students. These activities may be:

(i) Recreational activities (ii) Personal activities (iii) Domestic activities (iv) Vocational activities (v) Community, civic and social activities (vi) National activities (vii) Group activities (viii) Project work. Further, the types of these activities may be changed as per age levels of the students.

(g) Principle of Individual Differences

Organisation of the mathematics curriculum for each Class and each stage must cater to the needs of all the categories of students. It means that there should be some topics for mathematically gifted students as well as for other students also. Similarly, needs of students from rural and urban areas as well as students coming from different strata of society should also be kept in mind while formulating and organising the mathematics curriculum.

This suggests that there may be some optional topics or questions in the curriculum. Further, there should also be some flexibility for the students to learn at their own pace.

In addition to the above mentioned principles, there are different approaches which may be followed for organising a mathematics curriculum. They are as follows:

(a) Topical Approach

In this approach, a topic, once presented, must be completely exhausted in the same Class or at the same stage. This approach has many drawbacks which are given below:

- It does not take into account the maturity level of the students
- Topics, once completed, receive no attention at the later stages and, hence, there is a chance of getting them forgotten
- There is no chance of relating a particular topic to other topics or to other branches of mathematics
- Reading a topic for longer period of time may result in making the subject dull and uninteresting

Due to above drawbacks, this approach is usually discarded in mathematics. For a systematic study, a topic cannot be covered just in one Class or at one stage. For example, all types of numbers cannot be covered only in Class VI or Class VII or at Upper Primary stage. Students must be allowed to learn at their own pace.

(b) Concentric Approach

In this approach, each topic is divided into smaller sub-topics or units. These smaller units are selected in such a way that they can be arranged in ascending order following both the difficulty and the logical orders. Now, each year, one Unit from each of the topics are included. For example, one Unit from number system is included in Class VI, next Unit of the number system is included in Class VII and so on. Similarly, one Unit of Algebra is included in Class VI, next Unit of algebra in Class VII and so on. Similarly, the process is repeated for geometry, etc.

In this approach, difficulty as well as maturity levels of the children are taken care of. It also provides opportunities to relate one topic with the other topics.

However, following this approach, there is a possibility that none of the topics could be covered at the school stage. For example, if basic concepts of geometry are included in Class VI, angles are included in Class VII, triangles in Class VIII, Congruence of triangles in Class IX and similarity of triangles in Class X, then should we include circles in Class XI? Again, in which Class, we should include construction? In view of the above, this approach is also not appropriate for developing mathematics curriculum.

(c) Spiral Approach

In this approach, a topic is divided into a number of smaller independent units to be dealt with in order of difficulty, keeping in mind the mental capabilities of students. Thus, contrary to the topic and concentric approaches, in the spiral approach, a topic is covered in a number of classes or stages, in the order of difficulty. To begin with, elementary concepts related to the topic are presented in one Class and then gaps are filled in the next Class or later part of the same year and more gaps in other higher Classes.

For example, in the NCERT mathematics curriculum (NCF – 2005), linear equations in one variable are included in Classes VI, VII and VIII with varied difficulty levels. Then, they are extended to two variables in Classes IX and X. Similarly, geometry is included in different forms at different school stages, and so on.

This approach has the following advantages:

- It provides sufficient motivation for the students
- Students are able to appreciate the relevance and significance of the topic, they are learning
- It provides opportunities for revision of the fundamental concepts related to the concerned topic
- It provides opportunities to relate a topic with other topics in mathematics as well as with other disciplines
- It satisfies the psychological needs of the students

(d) Logical and Psychological Approaches

In the logical approach, each new idea, when introduced, should be anchored to the solid foundation of other well established ideas upon which it must stand to have a meaning. In other words, any new idea should not contradict the other well established ideas. For example, point, line and plane are considered as undefined terms and this fact is discussed in Class IX. However, in Classes VI, VII and VIII, concepts of point, line and plane are just explained, without any definition or otherwise. On the other hand, in the psychological approach, attempt is made to maintain interest of the students by the proper selection and use of materials, wherein the mathematical concepts are used over and over again.

A mathematician shall always desire that mathematics should be dealt with, in a strictly logical manner, while a classroom teacher may desire to deal it following a psychological approach, keeping in view his/her students in mind. Thus, for a curriculum organiser, it is essential to strike a balance between the two approaches. Some rigour may be sacrificed by the curriculum organiser for the sake of a better psychological treatment in mathematics.

EXERCISE 4.3

1. Answers to which questions provide a basis for designing a curriculum?
2. List the areas on which decision-making is necessary for designing a curriculum.
3. What are the two stages involved in designing a curriculum?
4. State and explain different principles of curriculum construction, with special reference to mathematics.
5. State and explain different principles of organising a mathematics curriculum.
6. Name different approaches that are followed for organisation of mathematics curriculum.
7. State some salient characteristics of different approaches for organisation of a mathematics curriculum. According to you, which approach is the best? Justify your answer.

4.5 Mathematics Curriculum for Different Stages of Schooling

At different stages of schooling, students are in different age groups. Therefore, treatment to mathematics curriculum at different stages must be of different types. Let us consider these stages one-by-one.

At the pre-primary stage, all learning occurs through playway activities rather than through didactic communication. Instead of rote learning of the number sequence, children need to learn and understand, in the context of small sets, the connection between word names and counting; and between counting and quantity. Making simple comparisons and classifications along one dimension at a time, and identifying shapes and symmetries are appropriate skills to be acquired at this stage. Encouraging children to use language to express one's thoughts and emotions freely, rather than in predetermined ways, is extremely important at this and at later stages.

At the primary stage, there is a need to develop a positive attitude towards mathematics. Mathematical games, puzzles and other recreational activities help in developing a favourable attitude and in making connections between mathematics and everyday mental activity. It is important to note that mathematics is not just arithmetic. Therefore, besides numbers and number operations, due importance must be given to shapes, spatial understanding, patterns, measurement and data handling. The curriculum must explicitly incorporate the progression that the learners make from the concrete to abstract, while acquiring concepts. Apart from computational skills, stress must be laid on identifying, expressing and explaining patterns, on estimation and approximation in solving problems, on making connections, and on the development of skills of language in communication and reasoning.

At the upper primary stage, students get the first taste of the power of mathematics through the application of powerful abstract concepts that compress previous learning and experiences. This enables them to revisit and consolidate basic concepts and skills learnt at the primary stage, which is essential from the point of view of achieving Universal mathematical literacy. Students are introduced to algebraic notations and their use in solving problems and in generalisations, to the systematic study of space and shapes and for consolidating their knowledge of measurement. The transition from arithmetic to algebra is both challenging and rewarding at this stage. Data handling, representation and interpretation form a significant part of the ability of dealing with information in general, which is an essential 'life skill'. The learning at this stage also offers an opportunity to enrich student's spatial reasoning and visualising skills.

At the secondary stage, students begin to perceive the structure of mathematics as a discipline. They become familiar with the characteristics of mathematical communication, carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions and proofs justifying propositions. These aspects are developed, particularly in the area of geometry. Students develop their familiarity with algebra, which is important not only in the application of mathematics but also within mathematics in providing justifications and proofs. At this stage, students integrate many concepts and skills that they have learnt into a problem-solving ability. Mathematical modelling, data analysis and interpretation taught at this stage can consolidate a individual and group exploration of connections and patterns. Visualisation and generalisation, and making and proving conjectures are also important at this stage.

The mathematics curriculum at the higher secondary stage should make the students realise a wide variety of mathematical applications and equip them with basic tools that enable these applications. A careful choice between the often conflicting depth versus breadth needs to be made at this stage. The rapid development of mathematics as a discipline and its range of applications forms an increase in the breadth of the coverage. Such increase must be dictated by the mathematical considerations of the importance of topics to be included. This stage is the launching pad from which the student is guided towards career choices, whether they apply for university education or otherwise. By this time, the students interests and aptitudes have been largely determined and mathematics education in these two years can help in sharpening students abilities.

EXERCISE 4.4

Write three main guiding principles for the mathematics curriculum for each of the following stages of schooling:

(a) Pre-primary stage (b) Primary stage (c) Upper primary stage (d) Secondary stage (e) Higher secondary stage. Explain each of these through some examples.

4.6 Some Highlights of the Mathematics Curriculum

As already stated, curriculum development is a dynamic and continuous process. Mathematics curriculum has undergone a number of changes from time to time in the last few decades. Every new mathematics curriculum has some unique features or highlights. For example, at the time of evolution of the so called modern or new mathematics curriculum, some highlights were as follows:

(i) Changes in the curriculum helped the students in meeting their needs.

(ii) Curriculum provided an understanding of mathematics for future change and development.

(iii) Curriculum provided application of mathematical structures and also metric and non-metric relations in geometry.

(iv) The curriculum materials involved experiences with appreciation of abstract concepts, the role of definitions, the development of precise vocabulary, thought, experimentation and proof.

(v) Curriculum provided experiences to explore the behaviour of numbers to describe new situations.

(vi) Curriculum provided certain unifying factors like set, function, etc. for different branches of mathematics and so on.

If we talk about the NCF – 2005, under which the present mathematics curriculum has been developed, it has the following highlights:

(i) Vision of School Mathematics:

- Children learn to enjoy mathematics rather than fearing of it
- Children learn important mathematics. Mathematics is more than formulas and mechanical procedures

- Children see mathematics as something to talk about, to communicate through, to discuss among themselves, to work on together
- Children pose and solve meaningful problems. Children use abstractions to perceive relationship, to see structures, to reason out things, to argue the truth or falsity of statements
- Children understand the basic structure of Mathematics: Arithmetic, Algebra, Geometry and Trigonometry, the basic content areas of school mathematics, all offer a methodology for abstraction, structuration and generalisation
- Teachers engage every child in class with the conviction that everyone can learn mathematics.

(ii) Main Goal of Mathematics Education is developing children's abilities of mathematisation. Basically it means that children should learn to think about any situation using the language of mathematics, so that the tools and techniques of mathematics can be used. This typically involves drawing pictures, choosing variables framing equations and arriving at a conclusion logically. For example, if the length of a rectangular field of area 400m^2 is twice its width, then it can be mathematised as $2x \times x = 400$, i.e., $2x^2 = 400$, where variable x represents width of the field (in metres).

(iii) Core Areas of Concern in School Mathematics include the following:

- A sense of fear and failure regarding mathematics among a majority of children
- A curriculum that disappoints both a talented minority as well as the non-participating majority
- Crude methods of assessment that encourage perception of mathematics as mechanical computations
- Lack of teacher preparation and support in the teaching of mathematics

EXERCISE 4.5

1. What do you understand by ability of mathematisation as the main goal of school mathematics education? Give some examples to explain the same.
2. Reflect on the vision of mathematics as envisaged in NCF – 2005.
3. What are the core concerns of mathematics education as stated in NCF – 2005? Do you agree with them? Give reasons for your answers.

4.7 Topics for Various Stages for Different Branches of Mathematics

We have a number of branches of mathematics at school stage, such as arithmetic, algebra, geometry, etc. Let us have a brief look at these.

Arithmetic: If we consider mathematics as the queen of sciences, then we can say that arithmetic is the queen of mathematics. It is arithmetic which provides base for other branches of mathematics. It deals with numbers and operations on them. Arithmetic paves the way for higher learning in mathematics. Even algebraic, geometrical and trigonometrical calculations make use of arithmetic. Strictly speaking, arithmetic can be divided into two parts, namely

(i) pure Arithmetic and (ii) applied Arithmetic

Pure Arithmetic involves the study of various types of numbers, such as natural numbers, whole numbers, integers, fractions, decimals, etc. Applied Arithmetic involves the study of topics, such as ratio, proportion, percentage, profit and loss, simple and compound interest, etc. These days, the above topics are generally studied under a separate heading, "Commercial Mathematics" or "Comparing Quantities".

Algebra: Mathematics is a language having its own symbols and grammar. This language has grown out of human needs. If we consider mathematics as a language, then Algebra is considered as the shorthand of the language of mathematics. It helps to make arithmetic more general. It is for this reason that algebra is considered as generalised arithmetic. It is essential to know algebra to study almost every branch of mathematics. It starts with the study of variables and constants and includes algebraic expressions, equations, factorisations, etc.

Geometry: Geometry is the study of space, figures and spatial relations. Points, lines and planes are building blocks of geometry. Geometry helps to measure lengths, areas and volumes of geometrical entities. It is also a study of relations, such as congruence and similarity between figures. These days topics related to areas and volumes are studied under a separate heading 'mensuration'.

Trigonometry: Trigonometry is the study of relations between sides and angles of triangles. Idea of similarity of figures is useful in the study of trigonometry. It has applications in finding heights and distances which cannot be directly measured.

Statistics: It is a allied discipline of mathematics that collects, analyses and interprets the information collected in the numerical form called data. Then the data is represented pictorially by bar graphs, histograms, pie charts, etc. and numerically by mean, median, mode, etc.

Probability: It is a branch of statistics which deals with the analysis of the occurrence of certain events which are useful for many practical purposes. In it, uncertainties are measured using special techniques and then predictions are made, e.g. weather forecasting, calculations of L.I.C. premiums, etc.

From the above, it can be seen that all the branches of mathematics are closely related and dependent on each other. Further, mathematics is a sequenced subject. Therefore, teaching of mathematics is a little different from the teaching of other subjects. In the case of other subjects, the student can make up the lost portion if he/she was absent at the time of teaching some topic of that subject. But in the case of mathematics, it is not possible. One cannot understand multiplication and division until he/she has learnt addition and subtraction properly. Profit and loss and simple interest cannot be understood by the students without proper understanding of percentages, and so on.

4.8 Pedagogical Analysis of Some Topics in Mathematics

Keeping the above in view, it is very important that topics in various branches of mathematics for different stages of schooling should be selected very carefully and judiciously after a thorough pedagogical analysis of the topics in arithmetic, algebra, geometry, trigonometry, etc.

It is imperative that the above selection of topics for different stages must be done keeping in view the following three points:

(i) There should be a logical flow.

(ii) There should be a psychological flow

(iii) There is a spiral approach.

The time available for teaching of mathematics in the schools should also be taken into consideration for selecting the topics for different stages of schooling. Assuming that a total of 180 periods (of 45 minutes duration) are available for mathematics teaching in a year, a tentative syllabus for the topic : 'Number System' for Upper Primary Stage (Classes VI - VIII) is suggested below:

CLASS VI

Number System **(Periods : 60)**

(i) Large Numbers: Consolidating numbers upto 5 digits, comparing larger numbers; Word problems on operations involving large numbers upto a maximum of 5 digits in the answer after all operations. This would include conversions of units of length and mass (from larger to smaller units), estimation of outcomes of number operations. Extending numbers up to 8 digits and approximation of larger numbers.

(ii) Factors and Multiples: Concepts of factors and multiples, divisibility rules by 2, 3, 4, 5, 6, 8, 9, 10 and 11 (All these through observing patterns); even and odd numbers; prime and composite numbers; coprime; prime factorisation of a number; HCF and LCM of two or more numbers by prime factorisation method. LCM by common division method also.

(iii) Whole Numbers: Natural numbers; Whole numbers; Properties of these numbers (closure, commutative, associative, distributive property; identity element); Representation on the number line; observing patterns.

(iv) Integers: How negative numbers arise?; connection to daily life situations; extending whole numbers to integers using number line; comparison of integers; addition and subtraction of integers.

(v) Fractions and Decimals: Review of fraction as a part of a whole; proper, improper and mixed fractions; equivalent fractions, comparison of fractions; addition and subtraction of fractions.

Review of the idea of a decimal fraction (i.e., a decimal), place value in the context of decimals; conversion of fractions to decimals and vice-versa (avoiding recurring decimals); Addition and subtraction of decimals.

CLASS VII

Number System (Periods : 50)

(i) Integers: Multiplication and division of integers; properties of integers (closure, commutative, associative, distributive property, identity element)

(ii) Fractions and Rational Numbers: Multiplication of fractions, fraction as an operator; reciprocal of a fraction, division of fractions; introduction to rational-numbers with representation on the number line, addition, subtraction, multiplication and division of rational numbers; representation of rational numbers as decimals; multiplication and division of decimals.

(iii) Exponents and Powers: Natural numbers as exponents. Following laws of exponents through patterns:

- $a^m \cdot a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$ $(m > n)$
- $(a^m)^n = a^{mn}$
- $a^m \cdot b^m = (ab)^m$

Writing large numbers in standard form.

CLASS VIII

Number System (Periods : 50)

(i) Rational Numbers: Consolidation of operations on rational numbers; properties of rational numbers (closure, commutative, associative, distributive property, identity element). representation of rational numbers on the number line; between any two rational numbers, there always lie an infinite rational number.

(ii) **Exponents and Powers:** Integers as exponents; laws of exponents extended to integral exponents; writing smaller numbers in standard form.

(iii) **Squares and Square Roots:** Concepts of a square and a square root; finding square roots by prime factorisation. Finding square roots by long division method of numbers (including decimals).

(iv) **Cubes and Cube Roots:** Concepts of a cube and a cube root; finding cube roots by prime factorisation method; Finding cube roots of perfect cubes (upto six digits) by observing their unit digits.

(v) **Generalised Form of Numbers:** Writing numbers, such as '*abc*' (where *a*, *b* and *c* are digits) in generalised form as $100a + 10b + c$; using this generalised form in solving some number puzzles and also in deriving some known divisibility rules (of 3, 9 and 11).

You can observe that how the ideas discussed in this Unit have been used in constructing the above syllabus. For example, using the spiral approach, the fractions introduced in Primary Classes have been reviewed in Class VI and then operations of addition and subtraction on them have been discussed. There after, the multiplication and division of fractions have been taken up in Class VII, and so on. In a similar manner, syllabus of mathematics involving other topics may be constructed for different stages of schooling. For that purpose, one may go through the present syllabi of mathematics prepared under the guidelines of NCF – 2005 by NCERT.

However, it will be worthwhile, if expected learning outcomes for different topics are also given with them. As suggested examples, learning outcomes for some topics are given below:

(A) Factors and Multiples

Learning Outcomes

After studying this topic, the student will be able to:

(i) understand the meaning of factors and multiples of numbers

(ii) understand the meaning of prime and composite numbers

(iii) find common factors and common multiples among two or more numbers

(iv) find the HCF and LCM of two or more numbers

(v) state tests of divisibility by 2, 3, 4, 5, 6, 8, 9, 10 and 11

(vi) apply the above knowledge in solving problems, etc.

(B) Fractions

Learning Outcomes

After studying this topic, the student will be able to:

(i) understand the meaning of a fraction

(ii) understand the meaning of numerator and denominator of a fraction

(iii) classify fractions into different types, such as proper, improper and mixed fractions; like and unlike fractions

(iv) compare two or more fractions and arrange them in ascending or descending order

(v) reduce fractions into the simplest or lowest form

(vi) perform four fundamental operations on fractions

(vii) apply the above knowledge in solving problems, etc.

As already discussed, a curriculum is much more than a syllabus. It also includes the textbooks prepared on the concerned syllabus. Therefore, some kind of pedagogical analysis of various topics is also necessary for developing the mathematics textbooks for various stages. As a suggested example, a topic namely 'Quadratic Equations' for Class X is being analysed, assuming that a total time of 15 periods is available for its teaching.

Topic : Quadratic Equations (15 Periods)

(A) Introduction/Motivation

It may start with the following example:

Example: "Length of a rectangular park is 1m more than twice its width. If the area of the park is 300 m^2, find the dimensions of the park".

Solution: Let the width be x metres. So, length = $2x + 1$. Thus, with the given conditions,

$$(2x + 1)\,x = 300$$

or $$2x^2 + x - 300 = 0 \qquad (1)$$

Here, equation (1) has been obtained with the help of previous knowledge of the students. However, students have not come across this type of equation so far, because they are aware with only linear equations in one or two variables. After obtaining this equation, the name 'Quadratic Equation' may be introduced to them and then they may be told that any equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ is known as a *quadratic equation.*

(B) Learning Outcomes

After studying this topic, the student will be able to:

- understand the meaning of a quadratic equation.
- identify quadratic equations from a given collection.
- solve a quadratic equation by factorisation
- solve a quadratic equation by completing the square, i.e., by the quadratic formula

- find the nature of the roots of a quadratic equation, using its discriminant
- solve word problems using quadratic equations

(C) Understanding / Identifying a Quadratic Equation

Check which of the following are quadratic equations:

1. $(x + 1)^2 = 2 (x – 4)$
2. $x^2 – 2x = (–2) (3 – x)$
3. $(x – 3) (x + 1) = (x + 5)^2$
4. $(x + 2)^3 = x (x^2 – 1)$
5. $x^2 + 3x + 1 = (x – 2)^2$
6. $(2x– 1) (x + 3) = (x + 4)^2$

(D) Solving Quadratic Equation by Factorisation

For this purpose, the concept of quadratic polynomial and its zeroes learnt earlier may be utilised. Also, the method of factorisation of a trinomial $ax^2 + bx + c$ by splitting the middle term 'bx' may be recalled through examples and the following types of equations may be solved and discussed:

1. $x^2 – 4x – 5 = 0$
2. $6x^2 – x – 2 = 0$
3. $2x^2 + x – 6 = 0$
4. $\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0$

(E) Solving Quadratic Equation by Completing the Square, i.e., by Quadratic Formula

This method may be explained by first taking a specific equation and the general equation as follows:

Specific Equation:

$$9x^2 – 15x + 6 = 0$$

or $$(3x)^2 – 2 \times 3x \times \frac{5}{2} + 6 = 0$$

or $$(3x)^2 – 2 \times 3x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6 = 0$$

or $$\left(3x-\frac{5}{2}\right)^2 - \frac{25}{4} + 6 = 0$$

or $$\left(3x-\frac{5}{2}\right)^2 = \frac{25}{4} - 6$$

or $$\left(3x-\frac{5}{2}\right)^2 = \frac{1}{4}$$

or $$3x-\frac{5}{2} = \pm\frac{1}{2}$$

So, $3x-\frac{5}{2}=\frac{1}{2}$ or $3x-\frac{5}{2}=-\frac{1}{2}$

i.e., $x = 1$ or $x=\frac{2}{3}$

General Equation:

$$ax^2+bx+c=0$$

or $$a^2x^2+abx+ac=0$$

or $$(ax)^2+2ax\times\frac{b}{2}+ac = 0$$

or $$(ax)^2+2ax\times\frac{b}{2}+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+ac = 0$$

or $$\left(ax+\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2-ac$$

or $$\left(ax+\frac{b}{2}\right)^2 = \frac{b^2}{4}-ac$$

or $$\left(ax+\frac{b}{2}\right)^2 = \frac{b^2-4ac}{4}$$

or $$ax+\frac{b}{2} = \pm\sqrt{\frac{b^2-4ac}{4}}$$

or $$ax+\frac{b}{2} = \pm\frac{\sqrt{b^2-4ac}}{2}$$

or $$ax = -\frac{b}{2}\pm\frac{\sqrt{b^2-4ac}}{2}$$

or $$ax = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1)$$

At this stage, students may be introduced to equation (1) in the form of *quadratic formula*. Then, the equations may be solved using this formula. However, for the pedagogical reasons, first those equations may be solved, which have been already solved using factorisation method. For example:

(Factorisation Method)

$$6x^2 - x - 2 = 0$$

or $$6x^2 - 4x + 3x - 2 = 0$$

or $$2x(3x - 2) + 1\,(3x - 2) = 0$$

or $$(3x - 2)\;(2x + 1) = 0$$

i.e., $3x - 2 = 0$ or $2x + 1 = 0$

i.e., $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

(Quadratic Formula)

$$6x^2 - x - 2 = 0$$

$$a = 6,\ b = -1,\ c = -2$$

So, $$x = \frac{+1 \pm \sqrt{1 - 4 \times 6 \times (-2)}}{2 \times 6}$$

or $$x = \frac{1 \pm \sqrt{49}}{12}$$

or $$x = \frac{1 \pm 7}{12}$$

i.e., $$x = \frac{1+7}{12} \text{ or } x = \frac{1-7}{12}$$

i.e., $$x = \frac{8}{12} \text{ or } x = \frac{-6}{12}$$

i.e., $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

At this stage, name of ancient Indian mathematician **Sridhara** who derived the quadratic formula, may also be mentioned.

(F) Discriminant and Nature of the Roots

After arriving at the quadratic formula, the role of the expression $b^2 - 4ac$ may be discussed in reference to the nature of the roots of a quadratic equations, explaining why it is termed as the 'Discriminant' of the equation.

(G) Solving Daily Life Problems

At the end, application of the quadratic equations may be highlighted by taking appropriate examples. In these examples, some of the questions from the ancient Indian texts may also be included. For example,

Out of a number of Saras birds, one-fourth of the number are moving about the lotus plants, $\frac{1}{9}$th coupled with $\frac{1}{4}$th as well as 7 times the square root of the number move on a hill. 56 birds remain in Vakula trees. What is the total number of birds? (Mahavira, around 850). In the same manner, other topics may be analysed for the purpose of developing the topics in the textbook.

EXERCISE 4.6

1. Describe how different branches of mathematics viz. Arithmetic, Algebra, Geometry, etc. are closely related and dependent on each other.
2. Give three examples to show that mathematics is a sequenced subject.
3. Explain with examples as to how it is important to have a balance between logical and psychological approaches in selecting the topics for different stages of school mathematics.
4. Explain, with examples, the advantages of keeping a spiral approach in selecting topics for different stages of schooling.
5. Construct syllabi of algebra and geometry for upper primary and secondary stages.
6. Pedagogically analyse the topic 'Relation and Function' for including it in a textbook at Higher Secondary Stage.
7. Write three appropriate definitions of a curriculum as per your opinion. Also, justify your answer.

8. Give some details of narrow aim and higher aim as envisaged in NCF, 2005.
9. Give some examples of each of the following from the present mathematics syllabus prepared under NCF–2005. (i) Logical Approach (ii) Psychological Approach (iii) Spiral Approach
10. What are the main highlights of NCF–2005 in regard to mathematics curriculum?
11. List some topics from different branches of mathematics, such as arithmetic, algebra, geometry, trigonometry, probability, etc. Also list learning outcomes for these topics.
12. Critically examine the mathematics syllabus prepared under NCF–2005 for different stages of schooling and list some gaps in it, if any.
13. Pedagogically analyse different topics in algebra, arithmetic, etc. and provide guidelines for developing textbooks for different stages of schooling.

Summary

In this Unit, we have discussed different objectives of curriculum, principles of curriculum designing and other aspects of curriculum at different stages of schooling with special reference to mathematics and its branches like arithmetic, algebra, geometry, etc. In particular, we have discussed and emphasised the following points:

- Curriculum is an educational programme or course of study to be run for reaching certain goal or destination.
- Curriculum is a sequence of planned activities and experiences available to the students based on certain predetermined objectives.
- Curriculum development is a continuous and dynamic process.
- Generally, curriculum is considered synonymous with the syllabus, but they are two different things.
- Curriculum has been defined by different authors in different ways. As per educational dictionary, it is defined as the total structure of ideas and activities developed by an educational Institute to meet the needs of the students and to achieve educational aims.
- A curriculum includes content, pedagogy, systematic characteristics and assessment.
- Mathematics is a self contained independent discipline with its own language and structure.
- The Education Commission (1964-66) was of the view, that if we want to survive as a Nation in the present competitive technological World, the proper foundation of mathematics education must be laid in the schools.

- The Education Commission (1964-66), NPE–1986, NCF–2000 as also NCF–2005 have agreed that mathematics as a subject must be compulsory up to secondary stage in India.
- As per guidelines of NCF–2005, the main goal of mathematics education is to build abilities of mathematisation in students.
- NCF–2005 has suggested two types of aims for mathematics education. They are:

  (i) Narrow Aim: Development of basic skills which include development of useful capabilities, particularly related to the numeracy, numbers, number operations, measurement, decimals and percentages.

  (ii) Higher Aim : Development of mathematical thinking which includes development of child resources to think and reason mathematically, to prove assumptions to logical conclusions and to handle abstractions.
- The curriculum designing involves the following two stages:

  (i) Curriculum construction and (ii) Curriculum organisation.
- Curriculum construction as well as curriculum organisation are done on the basis of various principles.
- By curriculum construction, we mean the selection of appropriate topics to be included in the curriculum.
- Curriculum is organised by maintaining a proper conceptual sequence and continuity among various topics.
- Different approaches, such as (i) topic approach, (ii) concentric approach (iii) spiral approach, (iv) logical approach, (v) psychological approach may be followed in organising a curriculum.
- Spiral approach mixed with proper balance between logical and psychological approaches is preferred while organising a mathematics curriculum.
- Treatment to mathematics curriculum at different stages of schooling must be in accordance with age groups and maturity levels of the students.
- The present mathematics curriculum as well as the past mathematics curricula have some features or highlights, such as vision of mathematics etc.
- Topics for school mathematics curriculum must be carefully selected, keeping in view the following points:

  (i) There should be a logical flow,(ii) There should be a psychological flow and (iii) There should be a spiral approach.
- For constructing syllabi of mathematics for different branches, such as arithmetic, algebra, etc. for various stages of schooling, topics must be selected after a proper

pedagogical analysis. Also, necessary learning outcomes must be listed along with the relevant topics.

- Arithmetic, algebra, geometry, etc. are different branches of mathematics which are closely related and dependent on each other.
- Padagogical analysis of different topics in arithmetic, algebra, etc. is also necessary for the development of other materials, such as textbooks on the constructed syllabus.

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UNIT 5

APPROACHES AND STRATEGIES IN TEACHING AND LEARNING OF MATHEMATICAL CONCEPTS

5.1 Introduction

According to National Curriculum Framework – 2005, the main goal of mathematics education in schools is the 'mathematisation of the child's thought process'. Clarity of thought and pursuing assumptions to logical conclusions is central to the mathematical enterprise. There are many ways of thinking, and the kind of thinking one learns in mathematics is an ability to handle abstractions, and an approach to problem solving.

To achieve this goal, there is a need to develop the inner resources of a growing child. In developing child's inner resources, the role that mathematics plays is mainly developing logical thinking skills in a child. Since mathematics helps in training and disciplining the mind and also in developing the power of thinking and reasoning, it is a mental tool for training and exercise of intellectual functions. Besides being an independent subject of study, it has its applications in other branches of knowledge. Mathematical skills and their applications form an indispensable tool in our daily life.

Mathematics is a complex system of concepts and not merely a collection of facts. The pupil should be helped to have a carefully chosen set of mathematical experiences through teaching in order to help him/her to form a new concept. If the concepts in mathematics at the foundation level are clear, interest and enjoyment in mathematics become automatic. The clarity of fundamental concepts and procedures also helps the learner in mastering difficult concepts of higher order. Thus, there is a necessity of teaching more enriched mathematics and in a better way.

In this Unit, we will discuss different teaching-learning strategies which can be used in teaching mathematical concepts.

Learning Objectives

After studying this Unit, student-teachers will be able to:

- explain the meaning and nature of a concept
- understand the general process of concept formation and concept assimilation
- understand the formation of mathematical concepts
- use following strategies for teaching mathematical content:
  - (i) activity based method
  - (ii) heuristic method
  - (iii) inductive – deductive method
  - (iv) problem solving method
- explain the formulation of conjectures and generalisations through illustrations
- differentiate between the teaching of mathematics and teaching of science.

5.2 Meaning and Nature of a Concept

The term concept is a multi-ordinal, i.e., it takes on a wide variety of meanings in different contexts and it is important to recognise that there are differences between the vague meanings given to the term by the layman and the more precise technical meanings needed by the educational researchers. For a layman, concept is synonymous with idea, notion and thought. For the researcher, this vague and general set of meanings is useful only as a starting point, but requires finer analysis.

Dictionary meaning of the word 'concept' is (i) a thought, an opinion, (ii) a mental image of a thing formed by generalisation from particulars, also an idea of what a thing in general is to be.

Concept is also a generalised idea about some objects, persons, or events. It stands for a general class and not for a particular object or event. It is a common name given on the basis of similarities or commonness found in different objects, persons or events of a class. In one sense, it is the general mental image of the subjects, events, experienced or perceived earlier.

Piaget labels concept as 'schema' and defines it as 'internalised mental behaviour of the child'. According to him schema are the ways of perceiving, understanding and thinking about the World. Piaget was primarily interested in those concepts that might broadly be called 'scientific', such as relationship of size and weight, physical changes, constant and variable, action and nature of living things etc. According to him, basic process of conceptualisation is intellectual development of the child. The child constantly organises his actions or operations on the experiences from the environment, whereas he or she at the same time, adapts to the environment by the process of assimilation and accommodation. According to Piaget, the process by which the schema changes is called accommodation. But at the same time the child does not want to give up an old schema in response to one or two disconfirming experiences and the schema influence the interpretation of experience. Piaget named this process assimilation.

Some other definitions of 'concept' are given as follows:

(i) A concept is an inferred mental process.

(ii) Concept has a common name of a class of objects, events or processes.

(iii) Concept is a mental construct or mental image of the object.

(iv) Concept is an abstraction of inferred experiences or information processing.

(v) Concepts in different disciplines treated as generalisations under which fragments of knowledge can be encapsulated.

Concepts are a kind of subject matter and we know that mathematics is a complex system of concepts and mathematical concepts are only one class of concepts. They are generalisations about the relationships among certain kinds of data. Development of concepts is basic in mathematical learning capacity since by means of concepts, other kinds of subject matter are learnt. In the next section, we will discuss, how these concepts are formed or developed, i.e., the process of concept formation and concept assimilation.

5.3 General Process of Concept Formation and Concept Assimilation

Concept formation is a type of discovery learning involving underlying psychological processes, such as discriminative analysis, abstraction, hypothesis generation and testing and generalisation.

Concepts are among the first things learnt by young children. They mostly learn concepts through concept formation method.

In the concept formation, children observe many examples of the object about which the concept will be formed, follow all the processes stated above and finally generalise to arrive at the concept. This is how the children formulate the concepts of cat, dog, ball, etc. Here, the observation of the objects will be first time in life.

Older children of schools as well as adolescents mostly acquire new concepts through a process of concept assimilation. *Concept assimilation is typically a form of meaningful receptive learning*.

In concept assimilation, definition of a concept is initially presented followed by exemplifying and paraphrasing it. It also involves various active cognitive operations. The learners go through the same process of differentiating hypothesis, generating and testing, and generalising before new meaning emerges.

Concept formation provides students with an opportunity to explore ideas by making connections and seeing relationships between items of information. This approach can help students to develop and refine their abilities to recall and discriminate among key ideas to see commonalities and identify relationships, to formulate concepts and generalisations, to explain how they have organised data and to present evidence to support their organisation of data involved.

Concept formation lessons can be highly motivational because students are provided with an opportunity to participate actively in their own learning. In addition, the thinking process involved helps them create new and expanded meaning of the World around them as they organise and manipulate information from other lessons and contexts in new ways. The teacher is initiator of the activity and guides students as they move co-operatively through the task.

In this instructional approach, students are provided with data about a particular concept. These data may be generated by the teacher or by the students themselves. Students are encouraged to classify or group information and to give description labels to their groupings. By linking the examples to the labels and by explaining the reasoning, the students form their own understanding of the concept.

The first phase of Inductive Thinking Model given by Hilda Taba is an example of a concept formation strategy. In this model, students group examples together on some basis and form as many groups as they can. Each group illustrates a different concept.

According to Hilda Taba, concept formation process involves three steps (i) identifying and enumerating data that are relevant to the problem (i.e., observation) (ii) grouping these data according to some basis of similarity (i.e., categorisation) (iii) developing categories and labels for the groups (i.e., conclusion). To engage students in each of these activities, Taba invented teaching moves (steps) in the form of questions. These eliciting questions are matched to particular types of activities. For example, the question, "What did you see?" might induce the students to enumerate a list. The question "What belong together?" is likely to cause students to group those things that have been listed. The question, "what would we call these groups?" would be likely to induce students to develop labels or categories. The purpose of this strategy is to induce students to expand the conceptual system with which they process information.

5.4 Formation of Mathematical Concepts

The following illustrations suggest how concepts are formed in mathematics as a instructional strategy.

Example 1

Concept: Prime Numbers

Step 1: Identifying and Enumerating Data That are Relevant to the Problem (i.e., Observation)

TEACHER : List some numbers less than 100 on the blackboard.

RANI : 10

RAVI : 25

SAVITA : 43

TABISH : 12

ATIYA : 19

ZEBA : 26

NATALIA : $\frac{2}{3}$

TEACHER : $\frac{2}{3}$ is different, isn't it?

ZAINAB : 39

SAVITA : 99

RASHMI : 97 and so on.

Note: Teacher writes all these numbers on the black board (say 10, 25, 43, 12, 19, 26, 39, 99, 97, $\frac{2}{3}$, …)

Step II: Grouping Those Items According to Some Basis of Similarity (i.e., Categorisation)

Now, the teacher will classify these numbers as in Table I and Table II given below:

Table I

43, 19, 11, 23, 53, 13, 29, 31, 47, 97, 17, 37, 41, 83, 71,…

Table II

10, 25, 12, 26, 39, 99, 54, 57, 81, 8, 15, 21, 27, 30, 32, 42, 45, …

TEACHER: Identify the special features of numbers in Table I, which are different from those of the numbers in Table II.

STUDENTS:

- These numbers are not divisible by 2
- These numbers are not divisible by 3
- These numbers are not divisible by 5
- These numbers are not divisible by 7

TEACHER: Are they divisible by any common number ?

STUDENTS: Yes, they are divisible by 1

TEACHER: Are they divisible by any other number?

STUDENTS: No

- They are not divisible by 9
- They are not divisible by 10
- They are not divisible by 12

Then the teacher asked, with what number are they divisible? Are they divisible by themselves?

STUDENTS: Yes, they are divisible by themselves.

Step III: Developing Categories and Labels for the Groups (i.e., Conclusion).

TEACHER: What did you conclude about the numbers in Table I? What are their special features?

STUDENTS: The numbers in Table I are divisible by 1 and themselves only. They are not divisible by any other number.

TEACHER: How many divisors each number in Table I consists?

STUDENTS: Each number has two divisors, 1 and the number itself.

TEACHER: These numbers are known as prime numbers. Can anybody define prime numbers?

STUDENTS: Yes, prime number is a number with exactly two divisors.

TEACHER: What about numbers in Table II?

STUDENTS: These numbers are different from the numbers in Table I in the sense that each of them is divisible by 1, itself and one or more other numbers.

TEACHER: Explain a little more.

STUDENTS:

- 10 is divisible by 1, 10, 2 and 5
- 25 is divisible by 1, 25 and 5
- 12 is divisible by 1, 12, 2, 3, 4 and 6

TEACHER: The numbers in Table II are known as composite numbers. Now, can you define prime and composite numbers?

STUDENTS:

- The number which is divisible by only two different numbers 1 and itself or the number which cannot be divisible by any number other than 1 and itself is known as a prime number.
- The number which is divisible by some number other than 1 and itself is known as a composite number.

TEACHER: What can you say about the number 1? Is it a prime number ?

STUDENTS: No !, since it is divisible by 1 and itself (which is again 1). Thus, it is not divisible by two different numbers. So, 1 is not a prime number.

TEACHER: Can we define these prime and composite numbers in terms of factors?

STUDENTS: Yes, a prime number has only two factors 1 and the number itself, while a composite number has more than two factors.

TEACHER: Can we extend this concept for numbers greater than 100, e.g., 102 or 107?

STUDENTS: Yes, 107 is a prime number since it has only two different factors 1 and 107 while 102 has factors 2,3,6,17, etc., apart from 1 and 102. Therefore, it is a composite number.

Students will be asked to give examples of prime numbers and that of composite numbers.

Examples of prime numbers: 5, 7, 13, 29, 31, 47, 19, 41, 71, 89, 23, …

Examples of composite numbers: 4, 8, 9, 10, 25, 12, 26, 39, 99, 54, 57, 35, 56, 98, 119, 121, …

Example 2

Concept: Quadrilateral

Step 1: Identifying and Enumerating Data That are Relevant to the Problem (i.e., Observation)

TEACHER: Let us start drawing different geometrical figures on the blackboard.

The students come to the blackboard one by one and draw the following figures:

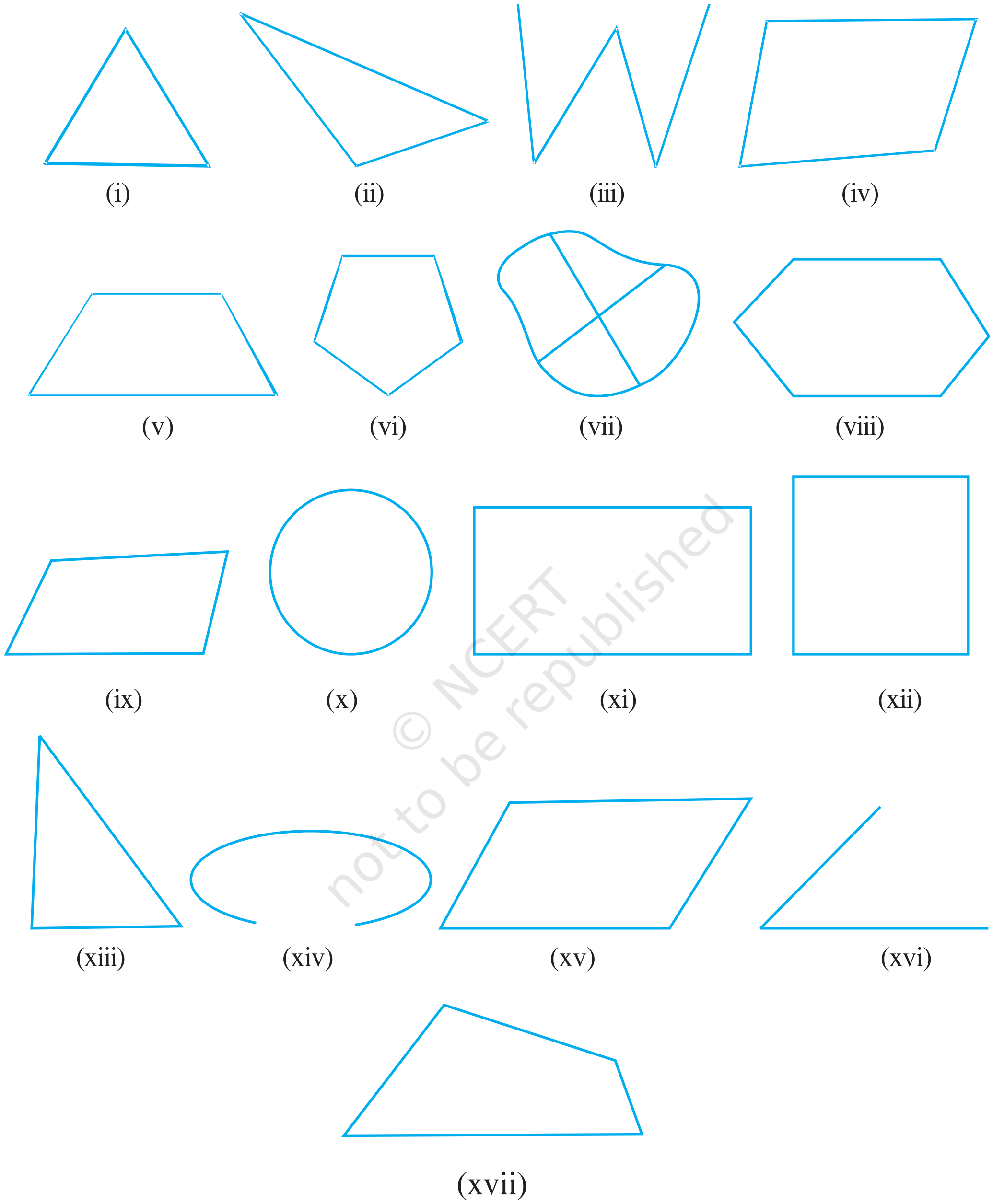


Fig. 5.1

Step II: Grouping These Figures According to Some Basis of Similarity (i.e., Categorisation)

Now, the teacher will classify these figures in Table I and Table II as given below:

Table I

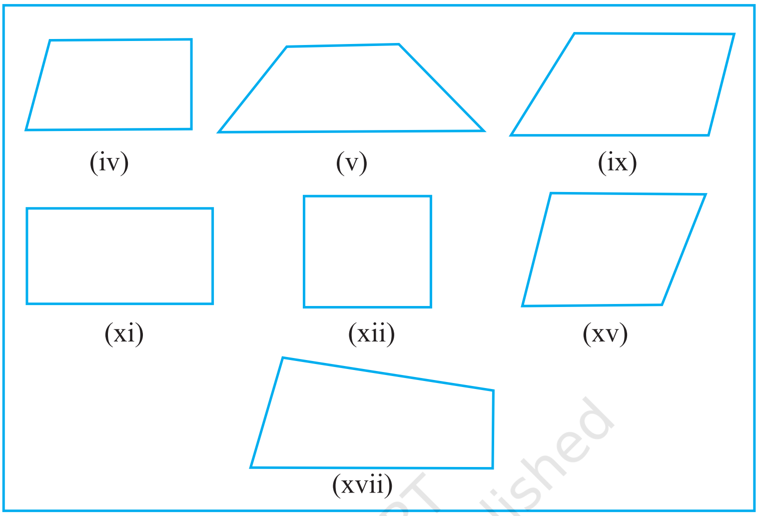
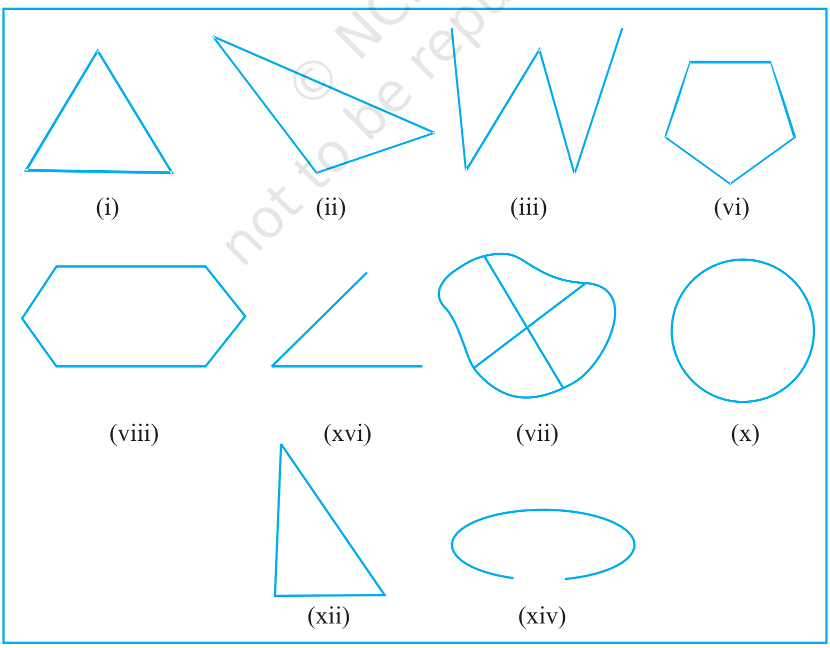


Table II



TEACHER: Identify the special features of figures in Table I, which are different from the figures in Table II.

STUDENTS:

- All the figures in Table I are closed.
- Measures of angles are 90°.

TEACHER: Are the measures of all the angles in each figure 90°?

STUDENTS: No, there is no specific relation among the angles in each of the figures.

TEACHER: Do you observe any other thing in these figures?

STUDENTS: Yes, each of them contains four equal sides.

TEACHER: Are the measures of all the four sides equal in every figure in Table I ?

STUDENTS: No, there is no specific relation among the lengths of the sides in each of the figures.

TEACHER: What about pairs of opposite line segments? Is there any relationship among them?

STUDENTS: They are parallel.

TEACHER: Are all the pairs of opposite sides parallel?

STUDENTS: No, in some cases one pair of opposite line segments is parallel, in other cases both the pairs are parallel, but in some cases none of the pairs is parallel.

Students tell about the figures with one pair of opposite sides parallel, both pairs are parallel and no pair is parallel.

Step III: Developing Categories and Labels for the Groups (i.e., Conclusion).

TEACHER: What did you conclude about the figures in Table I? What are their special features?

STUDENTS: Each figure in Table I

- is closed
- has four sides

TEACHER: What can you say about these figures?

STUDENTS: All figures are made up of four line segments and are closed.

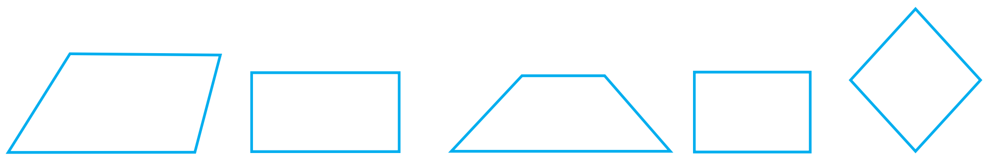
TEACHER: These types of figures are known as quadrilaterals. Thus, "A quadrilateral is a four sided closed figure."

STUDENTS: What about figures in Table II?

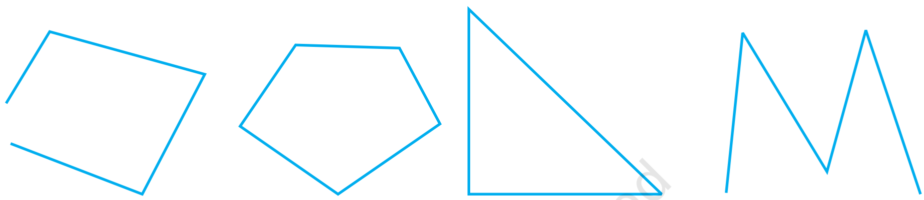
TEACHER: Definitely, none of them is a quadrilateral and some of them are not closed.

Now, the students will be asked to draw some examples and non-examples of quadrilaterals in their notebooks.

Examples:



Non-Examples:



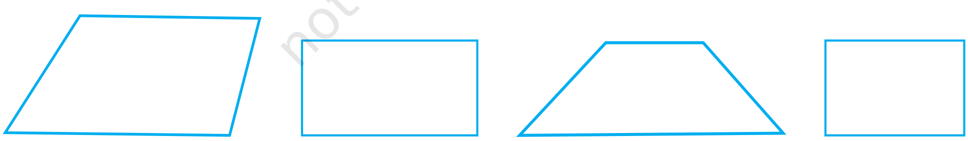
Example 3

Concept: Parallelogram

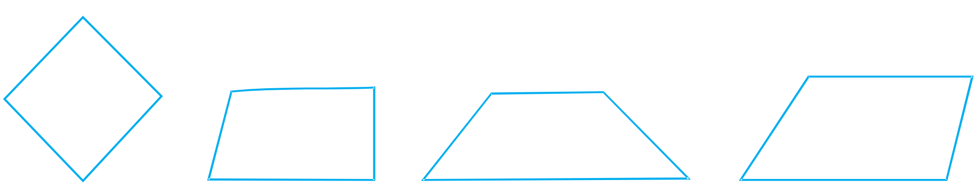
Step 1: Identifying and Enumerating Data That are Relevant to the Problem (i.e., Observation)

TEACHER: Let us start drawing some quadrilaterals on the black board.

STUDENTS: The students come to the blackboard one by one and draw the following quadrilaterals.



(i) (ii) (iii) (iv)



(v) (vi) (vii) (viii)

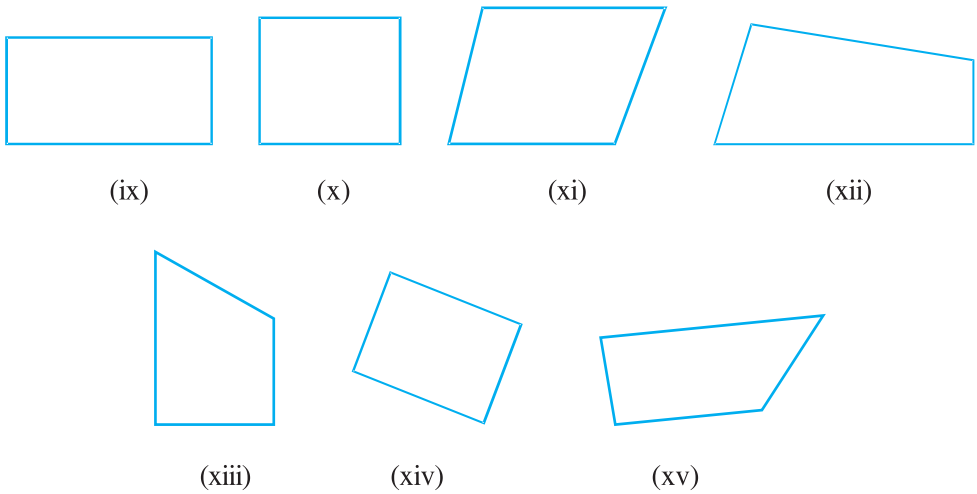


Fig. 5.2

Step II: Grouping These Figures According to Some Basis of Similarity (i.e., Categorisation)

Now the teacher will classify these quadrilaterals as in Table I and Table II

Table I: (ii), (iv), (v), (ix), (x), (xiv)

Table II: (i), (iii), (vi), (vii), (viii), (xi), (xii), (xiii), (xv)

TEACHER: Identify the special features of quadrilaterals in Table I, [which are different from the quadrilaterals in Table II].

STUDENTS: In these quadrilaterals:

- opposite sides are parallel
- measures of angles are 90°.

TEACHER: Are the measures of all the angles in each figure 90° ?

STUDENTS: No, in some quadrilaterals the measures of all the angles are 90°.

TEACHER: What about others?

STUDENTS: Opposite angles are equal.

TEACHER: What about pairs of opposite sides? Is there any relationship between them?

STUDENTS: Yes, they are parallel.

TEACHER: Are both the pairs of opposite sides parallel?

STUDENTS: Yes, both the pairs of opposite sides are parallel.

Step III: Developing Categories and Labels for the Groups (i.e., Conclusion).

TEACHER: What did you conclude about the figures in Table I? What are their special features?

STUDENTS: The figures in Table I:

- All of these figures are quadrilaterals.
- Both the pairs of opposite sides are parallel.

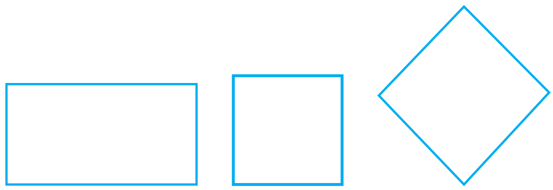
TEACHER: What can you say about these figures?

STUDENTS: All these quadrilaterals have both the pairs of opposite sides parallel.

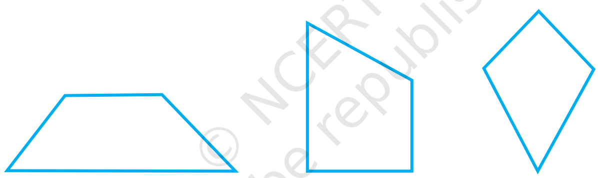
TEACHER: These types of quadrilaterals are known as parallelograms. Thus, "A parallelogram is a quadrilateral in which both the pairs of opposite sides are parallel".

Now, the students will be asked to draw some examples and non-examples of parallelograms in their notebooks.

Examples:



Non-Examples:



Example 4

Concept: Transpose of a matrix

TEACHER: Having learnt what a matrix is, today we shall learn the concept of 'transpose of a matrix'.

Transpose of a matrix is a matrix whose rows are columns of the given matrix.

Let us consider a matrix H $= \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$.

Now change, the corresponding rows into columns or columns into rows. What do you get ?

STUDENTS: I have got the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$$

TEACHER: A is known as transpose of matrix H.

STUDENTS: Can we say H is transpose of matrix A?

TEACHER: Yes, what can we conclude then?

STUDENTS: If transpose of matrix A is matrix B, then transpose of matrix B is matrix A.

TEACHER: Now, write the transpose of matrix $\begin{bmatrix} 0 & 1 \\ 8 & 3 \\ -2 & 4 \end{bmatrix}$

STUDENTS: $\begin{bmatrix} 0 & 8 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

TEACHER: Transpose of which matrix would be $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 6 \\ 3 & 7 & -8 \end{bmatrix}$?

STUDENTS: $\begin{bmatrix} 2 & 4 & 3 \\ 3 & -2 & 7 \\ -1 & 6 & -8 \end{bmatrix}$

TEACHER: Right. Are the following two matrices transpose of each other?

$$K = \begin{bmatrix} 2 & 0 \\ 1 & 7 \end{bmatrix}, \; L = \begin{bmatrix} 7 & 1 \\ 0 & 2 \end{bmatrix}$$

STUDENTS: No, they are not transpose of each other.

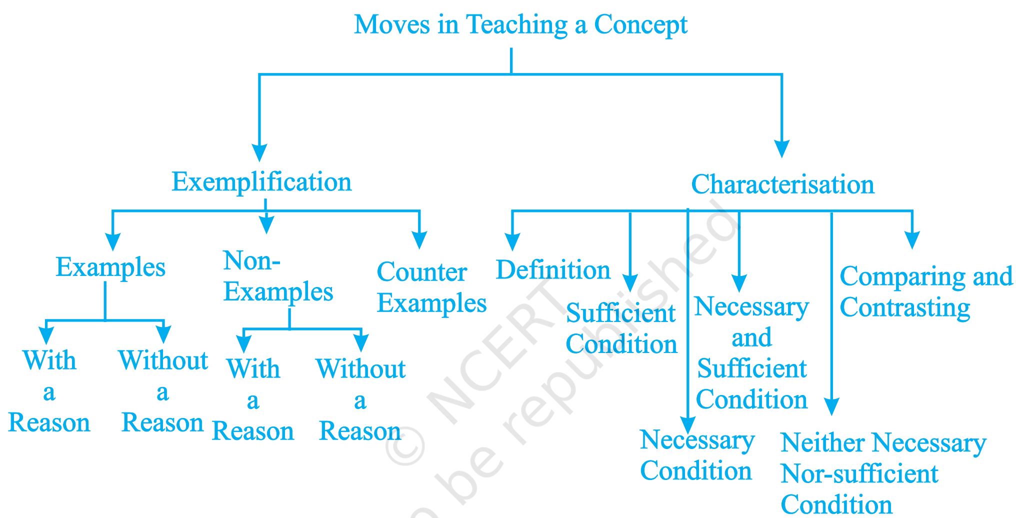
Today, we learnt about transpose of a matrix.

[Concept assimilation is a more appropriate strategy to teach relational concepts like transpose of a matrix, inverse of a matrix, congruency, etc. Relational concepts are concepts which are defined in relation to another concept. Transpose of a matrix exists only when a matrix is given.]

Moves in Teaching a Concept

Some concepts are taught deliberately and are usually learnt; others are learnt but not deliberately taught. That is, the teacher consciously takes time in class to teach some concepts. For others, the teacher simply uses a term that designates the concept; with repeated use the students acquire a concept designated by it. Thus, a teacher who has deliberately taught prime numbers might not take class time to teach a concept of composite numbers but would simply use the term and the students would probably make the correct inferences and acquire the concept.

During teaching a concept, if we analyse classroom dialogue or exposition, we can identify patterns of language usage—defining, giving examples, asserting, classifying, comparing or contrasting. We call them as *moves.* The moves are used by the teacher or by the students or by both. The purpose of the moves is to teach, present or clarify a concept. The following diagram shows a relation among the moves.



In teaching a concept, you name either objects denoted by the concept or objects not denoted by the concept; these are exemplification moves. Alternatively, you may mention characteristics or properties that enable students to find examples or non-examples; these are characterisation moves. Exemplification moves are of three kinds; examples, non-examples and counter examples. Counter examples are possible only in the context of a false generalisation. An example can be accompanied by a reason or not; the same can be said for a non-example also. Now, for characterisation moves, some are sufficient, some are necessary, some are both necessary and sufficient and some are neither necessary nor sufficient. Those that are both necessary and sufficient are definitional. Other characterisation, such as comparison and contrast are accomplished in terms of the characteristics of examples and non-examples. All these moves have already been discussed in this Unit, i.e., in concept formation and concept assimilation and we will discuss generalisation in detail at the end of the Unit.

5.5 Strategies for Teaching Mathematical Concepts

Educators have always been in search of more potential ways of instruction to help students learn effectively. In recent years, educators have suggested various directions for the improvement of teaching and learning. Beginning with demonstrations, enquiry, discovery method and problem solving techniques, educators have suggested constructivism in the classrooms as an interpretative process involving individual's constructions of meanings related to specific occurrences and phenomena. New constructions are built through their relations to prior knowledge and it is a pedagogic challenge for the teacher to focus on students' learning with understanding.

To learn mathematics, in constructivist approach, constructivism implies direct experience with mathematics as a process of knowledge generation in which prior knowledge is elaborated and changed on the basis of fresh meaning negotiated with peers and teacher. Some strategies of teaching mathematics which help the students in constructing their knowledge are:

- Activity based method
- Heuristic method
- Inductive-deductive method
- Problem-solving method

Now, we will discuss these strategies one by one.

5.5.1 Activity Based Method

This method is very important in teaching of mathematics due to the abstract nature of the subject. A phobia has been created in the minds of the children that mathematics is tough to learn. As a result, most of the students are not taking interest in the subject and it has become one of the main causes of student's failure in mathematics. No doubt, mathematics has inter-connected series of concepts and conceptual schemes, clarity of thoughts and logical conclusions is central to the mathematical enterprise. There are many ways of thinking, and the kind of thinking one develops in mathematics is an ability to handle abstractions. Even more importantly, what mathematics offers is a way of doing things: to handle and to solve mathematical problems, and more generally, to have the right attitude for problem-solving and to be able to attack all kinds of problems in a systematic manner. An important consequence of such requirements is that school mathematics must be activity-oriented.

Let's see how the concept of a fraction can be constructed through the use of activity based method or constructivist approach.

For this, teacher will draw a number of sets of parts of a whole as shown in the following figure:

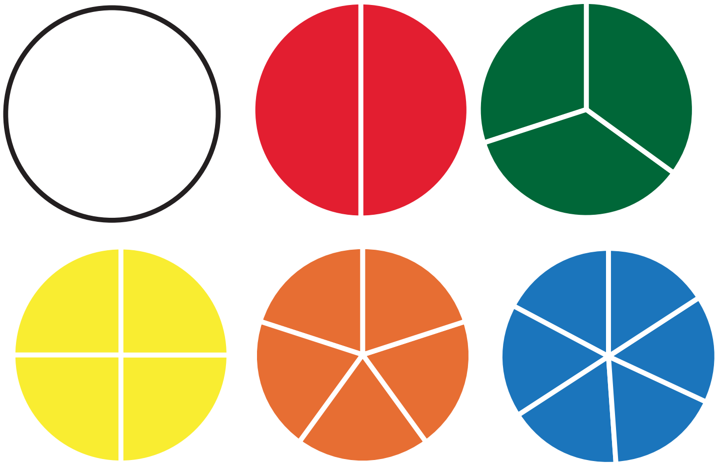


Fig. 5.3

Each set contains one whole (white), halves (red), thirds (green), fourths (yellow), fifths (orange), sixths (blue), and so on.

The teacher discussed fractions or the numerical relationships embodied in these parts, to assess the understanding of the students. After that, she pairs up students and passes out one set of pieces to each pair of students. She gives them a short time (5-10 minutes) to play with them as they wish and she watches them, to see what mathematical thinking is going on and asks students, how they went about organising the pieces, what they can tell her about the way the pieces work, etc. She asks them to name all the unit fractions (e.g., one-half, one-third, etc.), if white represents one whole and also why the pieces are named the way they are (e.g., thirds are called that because it takes 3 to make a whole). She supplements understanding by asking students to sketch circular models that correspond to the pieces.

Change the whole from white to other pieces, and then ask what unit fractions the remaining colours represent. (e.g., if red is one, a yellow is a half, a blue is a third, a purple is a fourth, a brown is a fifth and so on as shown in Fig 5.3. Students can get the idea by staking pieces on top of each other. At first, we just want the whole-part concept of fractions understood concretely. Supplement understanding by having students use colour squares to model fractions (e.g., $\frac{3}{5}$ may be modelled by showing 2 red squares and 3 blue squares. Here $\frac{3}{5}$ of these squares are blue. Do a variety of this type of examples. Ask students to make up their own).

Using white as a whole, count pieces by unit fractions (e.g., one-fourth, two-fourths, etc). Use both improper and proper fraction name – orally.

Repeat the above activities by writing the appropriate numerals and fraction symbols. Have students write the symbols next to the models. What does the denominator mean? What does the numerator mean?

Once the students get the concept of fractions, teacher will ask the students to compare two unit fractions by showing these pieces, e.g., which is greater, $\frac{1}{2}$ or $\frac{1}{3}$?, $\frac{1}{4}$ or $\frac{1}{5}$? Do a variety of these comparisons orally using the pieces. Then write symbol '>' as you do the comparisons over again. Teacher can make a table on chart paper/blackboard to organise students' responses, e.g.,

$$\frac{1}{2} > \frac{1}{3}$$

$$\frac{1}{3} > \frac{1}{4}$$

$$\frac{1}{4} > \frac{1}{5}$$

Teacher asks students: what pattern or rule can you see here? Write down the rule for unit fractions. Now teacher will ask the students to compare any two fractions with the same denominators, or any two fractions with the same numerators, by using the same approach as above.

After that the teacher will give the concept of equivalent fractions. She may ask students to use the parts of a whole of their set to show all the ways they can show a given unit fraction. For example, what parts (of the same colour) can you use to build $\frac{1}{2}$ (red). As students respond, write their findings in a systematic way using the equality sign (=), e.g., $\frac{1}{2} = \frac{2}{4}$. Write a string of equalities:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{?}{12}$$

$$\frac{1}{3} = \frac{2}{6} = \frac{?}{9} = \frac{4}{12} = \frac{5}{?} = \frac{?}{?}$$

$$\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{?}{16} = \frac{5}{?} = \frac{?}{?}$$

Have students look horizontally along the strings to see the pattern that numerators and denominators are same multiples. Also have students look vertically at fractions in a string to see the pattern that numerators are always the same part of the denominators. Have students extend patterns for more equivalent fractions and create their own strings using other unit fractions and the concept of multiple. Supplement understanding by doing a paper folding activity for showing $\frac{1}{2}$ and $\frac{1}{3}$.

Students can also build non-unit fractions from other fractions from their set of pieces. They use pieces to repeat the activity starting from non-unit fractions, such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{3}{5}$. Ask students to extend patterns to find further equivalent fractions.

Another concept that students can get from this set is, 'building common denominators'. They choose two (or more) fractions that can be built from the same unit fractions for example, $\frac{1}{2}$, $\frac{2}{3}$ can be built from sixths, $\frac{1}{4}$ and $\frac{1}{3}$ can be built from twelfth. Ask the students to see how many different pairs of fraction can be built from the same unit fraction. Write the symbols in a systematic way so that the students may see patterns of what common denominator means, e.g.

Fraction 1	Fraction 2	Common Unit Fraction	Equivalent Fraction 1	Equivalent Fraction 2
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{4}{6}$
$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{9}{12}$	$\frac{4}{12}$
$\frac{1}{6}$	$\frac{3}{4}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{9}{12}$
$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{6}{10}$	$\frac{5}{10}$

Students should be helped to see that the common denominator is the LCM of the denominators of the two fractions.

Now, we will see some more examples of teaching mathematics through 'activity based method'.

Activity 1

Topic: Verification of the algebraic identity $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

(i) Material required: Blocks.

(ii) Description of the activity

The following activity is performed to verify the identity:

$$(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

The formula for the identity $(a + b)^3$ can be easily derived from actual multiplication of binomials in the following manner:

Since $(a + b)^3$ can be written as product of $(a + b)^2$ with $(a + b)$

i.e., $(a + b)^3 = (a + b)^2 \times (a + b)$

Now we know that

$$(a + b)^2 = a^2 + 2ab + b^2$$

Therefore, by substituting the value of $(a + b)^2$ in the above expression, we may get

$$(a + b)^3 = (a^2 + b^2 + 2ab) \times (a + b)$$

This can be simplified further as follows:

$$(a + b)^3 = a \times (a^2 + 2ab + b^2) + b \times (a^2 + 2ab + b^2)$$

$$= a^3 + 2a^2 b + ab^2 + ba^2 + 2ab^2 + b^3$$

Next, we can arrange and combine the terms to get

$$(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

Since we derive our result mathematically, let us now try and understand this identity.

To understand this concept the teacher will divide the whole class into groups of five students, and then she distributes a packet of 27 blocks to each group. For the purpose of verification, let us start with an example of supposing our 'a' as '1', and 'b' as '2', i.e., $a = 1$, $b = 2$

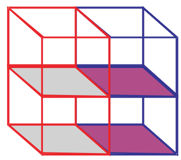
Then the teacher will ask the groups to find out the block which represent the *cube of 'a'* (i.e., cube of 1, obviously it will be 1block, since $a^3 = 1^3 = 1$), *cube of 'b'*, (i.e., cube of 2, obviously it will be 8 which means we will need 8 blocks in the form of cube of 2 such that there are 2 blocks each lengthwise, breadthwise and widthwise respectively).

(i) $a^3 = 1^3 = 1$ block

(ii) $a^2b = 1^2 \times 2 = 2$ blocks

Fig. 5.4

Next, the teacher asks the groups to find out blocks which represent $3a^2b$ and $3ab^2$ ($a^2b = 1 \times 1 \times 2 = 2$, i.e., we will need 3 pairs of 2 blocks each so that we get $3a^2b$). These shall be cuboids comprising of 2 blocks. Therefore, $a^2b = 2 = 2$ blocks. Next, we could see that identity involves terms like $3ab^2$, i.e., we will need to calculate ab^2. Therefore, $ab^2 = 1 \times 2 \times 2 = 4 = 4$ blocks, i.e., we shall be requiring 3 sets of 4 blocks each to represent $3ab^2$ (See Fig. 5.5).

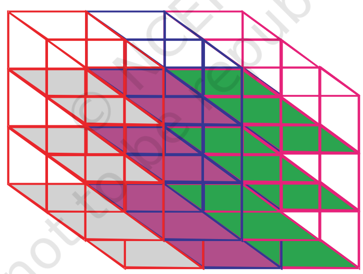


$ab^2 = 1 \times 2^2 = 1 \times 4 = 4$ blocks

Fig. 5.5

Now, teacher asks the students' group to combine all these blocks together. These blocks have pins at their ends so that these can be fixed one inside the other or can be joined / attached to each other easily.

Let us see the arrangement we obtain after joining them and then count the total number of blocks, which has made that arrangement(See Fig. 5.6).



$(a + b)^3 = (1 + 2)^3 = 3^3 = 27$ blocks

Fig. 5.6

Thus, we see that all blocks together constitute a cube of 27 blocks, i.e., 27 = cube of 3 (because it constituted a cube of length, breadth and width of side 3cm each).

$= (3)^3 = (1 + 2)^3$

Therefore, $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

Hence, the result is verified.

Activity 2

Topic: Concept of Operation on Integers

Material Required: 25 small square pieces of papers. These pieces are of red colour from one side and of blue colour on the other side.

Description of the Activity

Concept of operations (addition) on integers can be explained with the help of the following Activity:

The whole class is divided into small groups and each group is provided with a packet of integers, in which there are (say) 25 small square pieces of paper. These pieces are of red colour from one side and of blue colour on the other side. Let us suppose that red colour represents '+1' and blue colour represents '–1'.

Rules of the Activity

For positive integers, put the tokens from red side and for negative integers put the tokens from blue side. If the integers are of the same type (sign), then put the tokens in one row and count them otherwise put them in another row, i.e., positive in one row and negative in another row. Pairs of different colours will be picked and the remaining tokens will represent the answer (See Fig. 5.7).

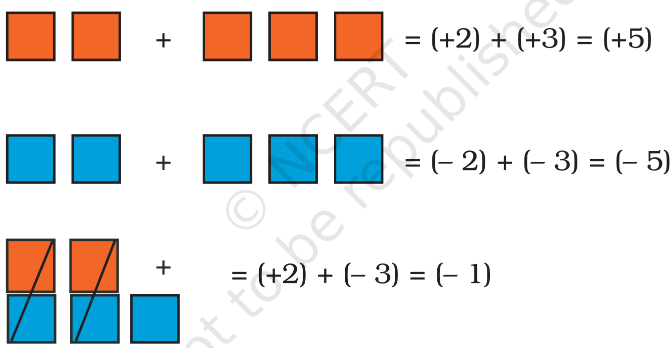


Fig. 5.7

Similar Activity can also be performed for understanding subtraction of integers.

Sign Rules for Multiplication

Let us describe the sign rule which runs as follows:

(i) (+) × (+) = +

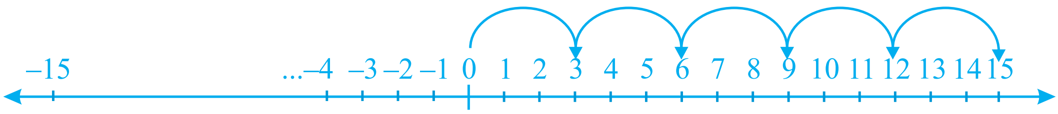
(ii) (+) × (–) = –

(iii) (–) × (+) = –

(iv) (–) × (–) = +

we begin with (i) by taking positive numbers on the real line say for example

(+3) × (5) = 15 (Why?)



(i) By (+3) × (5), we mean 3 units in the right direction taken 5 times giving 15, i.e., (+3) × (5) = 15

(ii) Let us illustrate the rule (+3) × (–5) = –15 (Why?)

Form (i), we have

$$3 \times 5 = 15$$
$$2 \times 5 = 10$$
$$1 \times 5 = 5$$
$$0 \times 5 = 0$$
$$-1 \times 5 = -5$$
$$-2 \times 5 = -10$$
$$-3 \times 5 = -15 \text{ (Why?)}$$

(iii) Similarly, we have (– 5) × 3 = –15

(iv) Again, to illustrate this rule by pattern, we proceed in the same way as done in above cases, i.e.,

$$(-3) \times 5 = -15$$
$$(-3) \times 4 = -12$$
$$(-3) \times 3 = -9$$
$$(-3) \times 2 = -6$$
$$(-3) \times 1 = -3$$
$$(-3) \times 0 = 0$$
$$(-3) \times (-1) = +3$$
$$(-3) \times (-2) = +6$$
$$(-3) \times (-3) = +9$$
$$(-3) \times (-4) = +12$$
$$(-3) \times (-5) = +15 \text{ (Why ?)}$$

If we look at the pattern, each time we find the product is decreasing by 5.

In this method, the students take part in the learning process; they investigate and explore, and thereby can actually 'find' mathematics themselves. Great mathematicians in past times and now do just that: they experiment and try this and that, make guesses and test them. With some guidance and with today's great learning aids, a child can rediscover mathematical truths in reasonable time (of course, mathematicians might have taken great deal of time to discover it.). Students should be made to think, instead of being provided with readymade answers to thought provoking questions.

Many children love mathematics in the beginning, but something during the school years kills that joy. Certainly we do need to teach the students those rules, but simultaneously, we also need to drill and practice these rules with them.

Merits

- It is a psychologically sound system of learning as it is based on the principle of learning by doing.
- Students gain stable and permanent knowledge and the understanding of concepts and facts are clear.
- It stimulates thinking of the children.
- It discourages memorisation and rote learning as knowledge is not imposed on the students.
- It facilitates more meaningful learning as the students actively participate in the learning process.
- It develops habit of hard work, patience, cooperation and self confidence among the students by keeping them busy to find out the required solution .
- It enables children to apply their knowledge in new situations.

Limitations

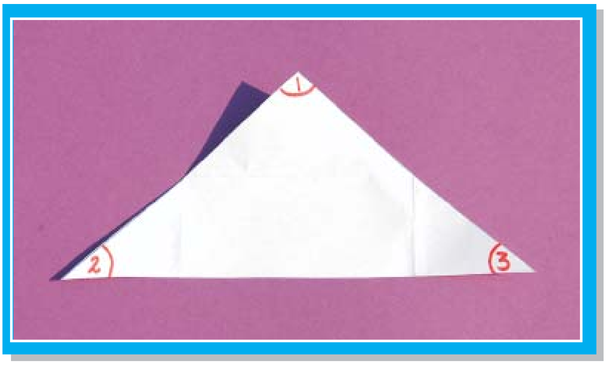
- It is a lengthy, time consuming method, and hence, it is difficult to cover the prescribed syllabus in time.
- To use this method, teachers and students have to do a lot of hard work and preparations.
- A lot of material resources is needed.

5.5.2 Heuristic Method or Discovery Method

We explain this method with the help of following example:

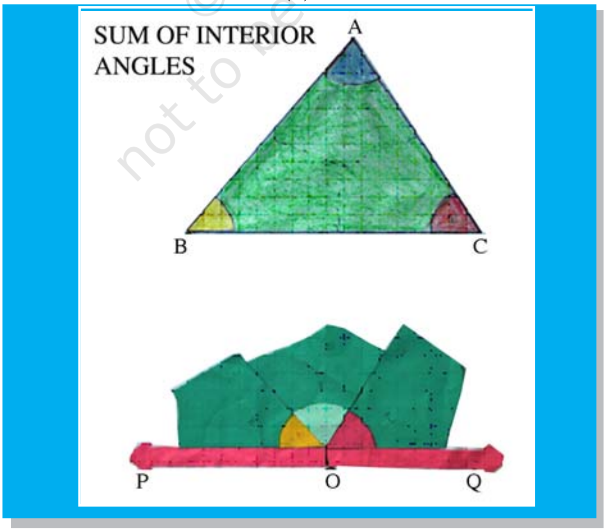
Topic: Angle Sum Property of a Triangle

The teacher will provide cut outs of different triangles to the students and ask them to find the sum of the angles.



(i)

(ii)



(iii)

Fig. 5.8

The students measure the angles, tabulate them and find the sum of the three angles, or the students can cut the three corners at the vertices and arrange them so that they form a straight angle. The students can fold the three corners and arrange them so that they form a straight angle or students can make three triangles of the same size and same shape, i.e., congruent triangles and try to put them in such a way that the three angles of different sizes form a straight line. The students have the freedom to use their own method of discovery and find the solution to the given problem. The teacher's role is to provide timely guidance and supplementary material and ask thought provoking questions to lead them in the right direction. This type of strategy is known as Heuristic Method or Discovery Method.

Another Example of Discovery Method is as Follows

Topic: Factorise $x^2 + 3x + 2$

Teacher will supply the material like

- Some square pieces of paper to represent the square of x
- Some rectangular pieces of paper of length x and breadth 1 unit to represent x
- Some unit squares of side one unit to represent 1

Teacher will ask students to try to represent above polynomial in the form of a square or rectangular shape with the help of given material. Students may get the figure (Fig. 5.9) as given below.

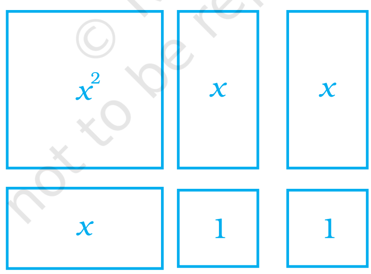


Fig. 5.9

By observing the above figure, they may conclude that the resulting figure is a rectangle of dimensions $(x + 2)$ and $(x + 1)$. From this Activity, they may conclude that the factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$. The Activity can be repeated for some other polynomials, such as, $2x^2 + 5x + 2$.

This method is more important from educational point of view because in this method students work like a researcher and solve the problems. By the use of this method, scientific and mathematical attitude can be developed in students.

Applications

In reality, heuristic method is not a method but an attitude which must be maintained during the teaching-learning process. The attitude is an essence of all the methods and must prevail all teaching and learning in the classroom.

Some of the common methods labelled as project method, problem-solving method, activity method, induction method (to be discussed later on) fall under the discovery approach.

Merits

- It develops a scientific and critical attitude in the students.
- It discourages memorisation and rote learning as the students discover rules/laws involved of their own.
- It arouses the spirit of enquiry in the students.
- It facilitates more meaningful learning as the students actively participate in the learning process.
- It develops habit of hard work among the students by keeping them busy to find out the required solution.
- It lays stress on individual practical work, careful observation and independent thinking which make the students self reliant.
- It gives understanding of concepts and facts.
- It is a psychological sound system of learning as it is based on the principle of learning by doing.

Limitations

- This method is feasible only with a highly resourceful teacher.
- It is a lengthy, time consuming method, and hence, it is difficult to cover the prescribed syllabus in time.
- In this method, the stress is more on skills than on acquisition of knowledge. This could lead to an imbalance in learning as knowledge is of secondary consideration.
- Not all students can cope with this type of learning.
- We cannot expect all the children to become discoverers and inventors.
- Sometimes wrong generalisations may be arrived at and thus a lot of time and energy may be wasted.

5.5.3 Inductive – Deductive Method

Inductive-deductive method is a combination of two approaches, inductive and deductive. First, we describe inductive approach with the help of some examples.

Example 1

Topic: Angle sum property of a triangle.

The teacher will ask students to draw a triangle in their notebooks and measure its angles.

She herself makes a table on the blackboard and asks the students to come to the blackboard and fill the table according to their measurements as shown below:

Triangle	Measure of angle 1	Measure of angle 2	Measure of angle 3
Tabish	70°	80°	30°
Rooshi	55°	72°	53°
–	–	–	–
–	–	–	–
–	–	–	–
–	–	–	–

After completing the table, teacher will ask the students to observe the table and try to generalise the relationship in the measures of the angles of the triangle. They can observe that sum of the measures of three angles of a triangle is 180°.

Example 2

Topic: $(a + b)^2 = a^2 + 2ab + b^2$

Teacher will ask the students to take up any binomial and multiply it by itself. Since at this stage, the students can perform the multiplication of a monomial by a monomial, monomial by binomial and binomial by binomial. She herself makes a table on the blackboard and asks the students to come to the blackboard and complete the table according to their calculations, as shown below:

Binomial	Binomial × Binomial	(Binomial)^2	Product
$(2x + 3y)$	$(2x + 3y) \times (2x + 3y)$	$(2x + 3y)^2$	$4x^2 + 12xy + 9y^2$
$(a + 2b)$	$(a + 2b) \times (a + 2b)$	$(a + 2b)^2$	$a^2 + 4ab + 4b^2$
$(3r + s)$	$(3r + s) \times (3r + s)$	$(3r + s)^2$	$9r^2 + 6rs + s^2$
–	–	–	–
–	–	–	–

After completing the table, teacher will ask the students to observe the table and try to generalise the emerging pattern. After observing the table, they can come to the conclusion that when we multiply a binomial by itself, we get sum of three terms – the square of the first term, square of the second term and twice the product of first and second terms. They can easily generalise that $(a + b)^2 = a^2 + 2ab + b^2$.

Example 3

Topic: Mean

The teacher will ask the students to write the marks obtained by them in the last terminal tests. She herself makes a table on the blackboard and asks the students to come to the blackboard and complete the table according to their calculations as shown below:

Name of the student	Marks in Mathematics	Marks in Science	Marks in Social Studies	Marks in English	Marks in Hindi
Joseph	10	8	6	9	7
Ajmal	9	7	Absent	8	8
Kavita	5	7	9	6	8
Shabnam	7	Absent	8	Absent	6
Shilpa	9	8	6	6	8
–	–	–	–	–	–

TEACHER: Students, here is a table showing marks of students in five subjects. Can you tell, who has overall best performance?

STUDENTS: Joseph, because he has got highest marks in mathematics (10).

TEACHER: But Joseph also has got least marks (6) in one subject.

STUDENTS: It is not possible to compare.

TEACHER: What can we do for that?

STUDENTS: We can add them and find out who gets the highest marks?

TEACHER: We can do that, but here two students were absent in some tests. By adding only can we get the correct conclusion? Will it be fair to compare the total of marks?

STUDENTS: No. It is not fair, as some of them attended five tests, some four and three.

TEACHER: Think about any other alternative to compare these marks.

STUDENT: One alternative would be to find the average.

TEACHER: Very good! How would you find the average of different students?

STUDENT: We add all the marks of a student and then divide it by 5.

TEACHER: Do you divide the total by 5 in every case?

STUDENT: No, for Joseph...

STUDENTS: For Joseph, we should add 10,8,6,9 and 7 and then divide it by 5, but for Ajmal, we will add 9, 7, 8 and 8 and then divide it by 4.

TEACHER: If there is a total obtained by adding 12 observations, how do we get the average?

STUDENT: The average is obtained by dividing the total by 12.

TEACHER: Then to obtain an average of n observations how to proceed?

STUDENT: To obtain the average of 'n' observations, n values have to be added and the total should be divided by 'n'.

TEACHER: Right. The procedure of finding average marks of some students, is as follows:

The average marks for Kavita will be $\left(\frac{5+7+9+6+8}{5}\right)=7$

For Shabnam the average marks will be $\left(\frac{7+8+6}{3}\right)=7$

TEACHER: How would you represent average $\overline{x}$.

STUDENTS: If we want to find out the average of n numbers, then it will be

$$\overline{x} = \left(\frac{x_1+x_2+x_3+...+x_n}{n}\right)$$

TEACHER: Right. This is the formula that can be used to find out average of 'n' numbers.

This average is also termed as 'mean'.

These are the examples of inductive approach. Inductive approach is based on the principle of induction. Induction is the process of arriving at a statement of generalisation by showing that it is true for many particular cases, and hence true for all the cases. So, the process of making transition from particular instances to generalisation is known as the induction. This approach proceeds from particular to general, from concrete instances to abstract rules and from simple examples to complex formula. This approach makes in children follow the subject matter with great interest and understanding. The knowledge attained by this approach becomes solid and durable, and, different mental powers of the child can also be developed.

Inductive method is most useful or suitable where

- New topics are introduced.
- Rules are to be formulated.

- Formulas are to be derived.
- Generalisation or laws are to be arrived at.

While selecting inductive method, a teacher should check whether it is possible to present sufficient number of particular cases as instances for the generalisation to be arrived at.

Steps in Inductive Approach

Inductive approach includes the following four steps:

1. Presentation of Specific Instances

Teacher presents many instances of generalisation before the students.

2. Observation

On observing various instances presented, the student would search for the commonalities among the instances.

3. Generalisation

After observing the instances presented, the teacher and children finalise the commonalities.

4. Verification

After deciding on commonalities, children verify it for other instances. In this way, children arrive at a generalisation.

Merits of Inductive Approach

- It is a scientific method because knowledge attained by this method is based on observations.
- Students gain stable and permanent knowledge with the help of inductive approach. On the basis of special instances, they practice the process of analysis and generalisation.
- It is a psychological method because many important principles of psychology are used in this method, i.e., child centred method.
- This method guides the students to work themselves, so this develops self-confidence in them.
- This method is appropriate for teaching mathematics as it is based on reasoning.
- It discourages the habit of cramming in students.

Limitations

- Principles/laws drawn by the use of this method are not always true.
- This is a very slow process. So, gaining knowledge by this approach costs more time and labour.
- To use this approach, teachers have to do a lot of hard work and preparations.

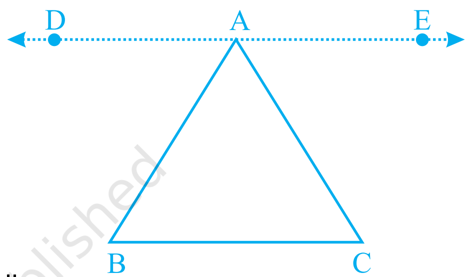
Deductive Approach

This approach is exactly the opposite of inductive approach. It is based on deductions. Here, the learner proceeds from general to particular, abstract to concrete and formula to example. In this method, laws, principles and formulas formulated on the basis of previous knowledge, experiments and examples are directly given to the students. Students themselves do not discover these laws and formulas. Using these laws and formulas, solutions of some problems are shown to them and then they are given more problems to solve with the help of those formulas. This approach is generally used in higher classes.

Deductive approach is complementary to inductive method. Now, we give some examples to describe the procedure of deductive method.

Example 1: Angle Sum Property of a Triangle

In a triangle, the sum of the three angles of a triangle is 180°.



Let ABC be a triangle.

Construct a line DE parallel to BC through A.

Here

$$\left.\begin{array}{l}\angle ABC = \angle BAD \\ \angle ACB = \angle CAE\end{array}\right\} \text{(alternate angles and } DE \parallel BC\text{)}.$$

But $\angle DAB + \angle BAC + \angle CAE = 180°$

So, $\angle ABC + \angle BAC + \angle ACB = 180°$.

Hence, the sum of the three angles of a traingle is 180°.

Example 2: Find out $(4a + 2b)^2$

$$\begin{aligned}(4a + 2b)^2 &= (4a + 2b)\,(4a + 2b)\\ &= 16a^2 + 8ba + 8ab + 4b^2\\ &= 16a^2 + 8ab + 8ab + 4b^2\\ &= 16a^2 + 16ab + 4b^2\end{aligned}$$

Merits

- Solving-problems with the help of pre-discovered formulas does not take much time. Students and teachers can apply this method frequently because by using this approach, mathematics work becomes very easy and efficient.

- It develops the learning power of students because students have to learn and retain so many formulas.
- This approach is very economical. It saves time and energy both for students and teachers.

Limitations

- Knowledge gained through this method is not permanent or stable.
- It encourages memorisation of facts which are soon forgotten and therefore, knowledge is rendered useless.
- This method is unnatural and is not psychological for the students who do not develop ability to appreciate abstract ideas in the absence of concrete examples.
- It fails to develop motivation and interest in the learners as the truths are not of much value to them.
- It fails to develop self-confidence and initiative in the students.

Combination of Deductive and Inductive Approach

Since inductive and deductive approach are complementary to each other, therefore, in the beginning, laws and formulas should be arrived at with the help of instances by the use of inductive method. After that, these laws and formulas should be derived with the help of deductive method. Therefore, inductive method is a forerunner and deductive method is its follower.

5.5.4 Problem-Solving Method

The basic purpose of education is to enable the child to adapt herself to life in the society which is full of problems. To be successful, one must be adequately equipped with proper reasoning and reflecting power. Therefore, it is very important that problem solving must be encouraged in school life. It is a basic skill needed by today's learners. Learners often learn facts and procedures with few ties to the context and application of knowledge. Problem-solving has become the means to rejoin content and application in a learning environment for basic skills as well as its application in various contexts. This implies that we must teach children how to think and reflect so that they are able to apply this to a vast number of varied problematic situations. Problem-solving ability enables them to find appropriate solutions to problems which confront them.

Now, the question arises, what is problem-solving? In the early nineties, problem-solving was viewed as a mechanical, systematic and abstract set of skills, such as those used to solve riddles or mathematical equations. These problems are based on logical solutions with a single correct answer. But under the influence of cognitive learning theories, problem-solving shifted to represent a complex mental activity consisting of a variety of cognitive skills and actions. It includes higher order thinking skills, such as visualisation, association, abstraction, comprehension, manipulation, reasoning, analysis, synthesis and generalisation, each to be managed and coordinated.

The process may be explained by the following example:

A picture is painted on a cardboard 8 cm long and 5 cm wide such that there is a margin of 1.5 cm along each of its sides. Find the total area of the margin.

Jyoti, a school teacher, said that while solving word problem, she used to emphasise that students should be able to visualise that problem, moreover, students can relate word problem to their daily life. In this way, they will understand the need of the subject and inculcate the interest in specific topics to be taught.

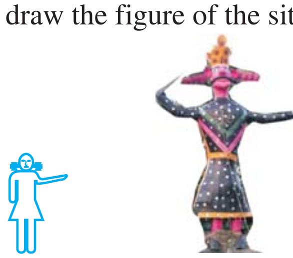
To solve this problem, she first of all, divides her class into groups and asks them to draw a picture of their own choice on the chart-paper of different measurements but do not to forget to leave a margin.

Next day all the groups have their painting; some of them were really beautiful. After admiring the paintings she asked them to try to measure the area of their paintings as well as the area of the margins to find the cost of lamination or framing their pictures. The class works in groups, she surprised to see that 80%-90% students obtained the correct solution.

Problem-solving is an instructional method or technique where by the teacher and pupils attempt by a conscious planned purposeful effort to arrive at some explanation or solution to some educationally significant difficulty. It is an individual or a small group activity, most efficient when done cooperatively with free opportunity for discussion. It increases a child's ability to think mathematically. The method of problem-solving is a method of thinking of analysing and of learning how to find the answer to a question or problem using known ideas. Learning through problem-solving is a progression, from known ideas to unknown ideas, from old ideas to new ideas and from the simple to the complex. It essentially results in an increased ability to think and generate ideas of mathematics. Problem-solving does not mean doing the block of exercises at the end of each Chapter or Unit.

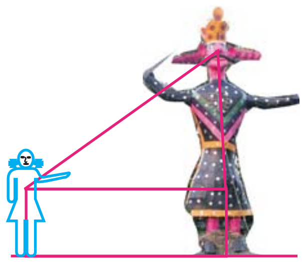
Another example of Problem-Solving Method is as follows:

In the days of Dussehra, Rani and Sunita went to see the Ramlila where they saw the statue of Ravana, which was very large. They wanted to find out its height. Next day, in their mathematics class, they asked their teacher the method to find out the height of Ravana's statue. The teacher asks them to draw the figure of the situation. They drew like this:



(i)

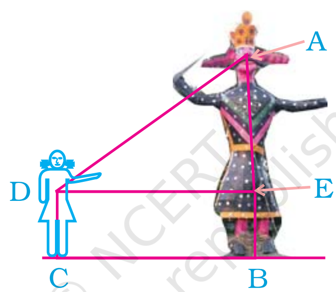
Teacher represents the situation mathematically, as follows:



(ii)

TEACHER: Can you see a right triangle in it? If yes, give some names to it.

STUDENTS: Yes, we can name it as follows:



(iii)

Fig. 5.10

TEACHER: What do you want to find out?

STUDENTS: Height of Ravana, i.e., AB

TEACHER: But in AB, you know EB, i.e., your own height, say 1 m, so what is left to find out?

STUDENTS: AE.

TEACHER: Now, write down the things we already know.

STUDENTS: It can be CB or DE, i.e., the distance between Ravana's statue and us, which is around 25 m.

TEACHER: In the previous study you learnt about trigonometric ratios. Recall the trigonometric ratio which involves AE and DE.

STUDENTS: $\tan \theta = \frac{AE}{DE}$

TEACHER: What is θ here? In this case, to see the statue, you have to lift your head, so the angle formed between your eyes and the head of the statue is known as angle of elevation. Here, θ represents the angle of elevation. If θ is 45º, then what is the value of tan θ?

STUDENTS: $\tan\theta = \tan 45^\circ = 1$.

TEACHER: Now to make the angle $\theta = 45°$, the observer has to move back and forth and identify the point with the help of a clinometer. Now, put the values and try to find the height of Ravana's statue.

STUDENTS: We have $\tan\theta = \frac{AE}{DE}$

Therefore, $\tan\theta = 1 = \frac{AE}{DE}$

$$1 = \frac{AE}{25}, AE = 25\,m$$

TEACHER: What is the height of the statue?

STUDENTS: AB = AE + EB

$$AB = 25m + 1m = 26m$$

Therefore, the height of the statue = 26m.

Steps in Problem-Solving Method

Individuals or groups can solve problems. But, group problem-solving is important because many diverse ideas are generated. There are different steps in problem-solving method, which we will discuss by taking an example.

Example: A mother is four times as old as her daughter, 3 years back she was 5 times as old as her daughter was then. What are their ages at present?

To solve this problem, the steps of problem-solving are as follows:

1. Recognising or Identifying the Problem

First of all, we sense the presence of a problem and then identify the problem. The investigation of solution must relate to the cause of the problem instead of its effect. After presenting the problem, teacher should give some time to students, to carefully go through the question. This provides the knowledge to every student to know what facts are given and what are to be determined, i.e., the students clearly understand the problem. Clarification of the problem involves making sure all students understand what the solution of the problem call for. This will include a review of the criteria for successful solution of the problem.

In the above problem, the student has to visualise that it requires him to find out the present ages of the mother and the daughter, both are unknown. Moreover, she has to understand the relationship between the unknowns.

(1) Mother's present age = 4 × the daughter's present age

(2) Mother's age 3years back = 5 × the daughter's age 3 years back.

2. Analysing the Problem

After presenting the problem, it should be analysed with the help of questioning method and with the help of students it should be made clear as to what is the relation between given facts. This develops the ability to analyse the problem in the students which is necessary to solve the problem successfully.

The student also has to visualise or ensure whether the information or data provided will be sufficient to solve the problems or not. This means that he has to analyse the problem situation.

3. To Search the Expected Mathematical Relations

After analysing the problem, the student comes to know which formula or laws are to be used to solve the problem? Here, students can decide which method is appropriate to find out the given solution. During this, students will generate as many ideas as possible. Ideas put forth will be encouraged with an emphasis on suspending judgment and criticism in order to encourage free flow of ideas and stimulate maximum output.

At this stage, the student first has to select suitable symbols for unknown quantities and then see relationships, if any, or in certain cases produce new relationships implicit in the data provided and ultimately to translate the action into symbols of the mathematical language to set up a suitable conditional equation or mathematical sentence. The concept of an equation as the expression of the structure of a problem situation is being used very frequently.

For the given example, the student will think in the way described below:

Representing mother's and daughter's present age as x and y respectively, we have

Mother's present age is four times the daughter's present age

$$x = 4y$$

Mother's age 3years back was five times her daughter's age 3 years back

$$x - 3 = 5(y - 3)$$

The required equations are

$$x = 4y \qquad (1)$$

$$x - 3 = 5(y - 3) \qquad (2)$$

4. To Find the Solution

At this stage, students solve the mathematical equation so obtained for the unknown variables. In this process, the student should check each step.

From equations (2) and (1), we get

$$4y - 3 = 5(y - 3)$$

or $$4y - 3 = 5y - 15$$

or $$4y - 3 + 3 = 5y - 15 + 3$$

or $$4y = 5y - 12$$

or $$4y - 5y = 5y - 12 - 5y$$

or $$-y = -12$$

or $$y = 12$$

From (1), $$x = 4 \times 12$$

or $$x = 48$$

Therefore, the mother's and daughter's present ages are 48 years and 12 years, respectively.

5. Check the Answer

After finding the answer, the results should be checked to correct the mistakes of computations, if any. The habit of checking the results should be developed in the students. Without this, it is difficult to find out the mistakes. This step involves relating the result obtained to the physical situation from which it comes. In fact, this step provides the student an opportunity to check her own thinking and the reasonableness of the answer. The student has to see whether the solution fits into the system of the problem situation. For the example given above, the process can be explained as follows:

At present

The mother's age is 4 times the daughter's age

$$48 = 4 \times 12$$

$$48 = 48$$

3 years back

The mother's age was 5 times the daughter's age

$$48 - 3 = 5\,(12 - 3)$$

$$45 = 5 \times 9$$

$$45 = 45$$

Sometimes, the students check their answers by substituting it in the equations like (1) and (2) and are satisfied with it, but this is not the correct way, since if somehow, the equations are wrongly formed, then, it will misguide the student as the answer will still satisfy the equations from which it was calculated. So, the best way is to relate the answer to the original situation to see whether its conditions are satisfied by the answer.

Merits

- This method is scientific in nature.
- It helps in developing good study habits and habits of reasoning, planning and independent working.
- It helps to improve and apply knowledge and experiences.
- It stimulates thinking of the child.
- Children learn how to act in new situations.
- It helps to develop group feeling while working together.
- It helps to verify an opinion and satisfies curiosity.
- This method is based on the principle of learning by doing.
- It develops qualities, such as patience, cooperation and self confidence.

Limitations

- This is not suitable for lower classes since small children do not possess sufficient previous knowledge to solve the problems and, therefore, fail to participate in discussions. They may not have developed vocabulary to understand the language of the problem.
- It is a time consuming process. Teachers find it difficult to cover the prescribed syllabus.
- To follow this method, committed teachers are required.
- Lack of interest and motivation on the part of the students can lower the effectiveness of this method.

5.6 Formulation of Conjectures and Generalisations through Illustrations

Suppose your brother has paid the electricity bill of your house for the month of September, 2010. The bill for October, 2010, however, claims that the bill of September has not been paid. How will you disprove the claim made by the electricity department?

You will have to produce a receipt proving that your September bill has been paid. You can see that this example shows that in our daily life, we are often called upon to prove that a certain statement of claim is true or false. However, we also accept many statements without bothering to prove them. But, in mathematics, we only accept a statement as true or false (except for some axioms only) if it has been proved to be so, according to the logic of mathematics. In fact, proofs in mathematics have been in existence for thousands of years, and they are central to any branch of mathematics.

Now, we shall try to explain the meaning of mathematically acceptable statements. A statement is a sentence which is not an order or an exclamatory sentence and, of course, a statement is not a question; for example "What is the colour of your hair?" is not a statement, it is a question. "Please go and bring me some water", is a request or an order, not a statement, however, "The colour of your hair is black", is a statement.

In general, a statement can be one of the following

- True
- False
- Ambiguous

For Example

State whether the following statements are always true, false or ambiguous.

(1) There are 8 days in a week.

(2) It is raining here.

(3) The sun sets in the West.

(4) Kanchan is a kind girl.

(5) The product of two odd integers is even.

(6) The product of two even natural numbers is even.

Solution

(1) This statement is false, since there are seven days in a week.

(2) This statement is ambiguous, since it is not clear where 'here' is.

(3) This statement is always true. The sun sets in the West no matter where we live.

(4) This statement is ambiguous, since it is subjective. Kanchan may be kind to some but not to others.

(5) This statement is false. The product of two odd integers is always odd.

(6) This statement is true. However, to justify that it is true, we need to do some work.

Mathematical statement cannot be ambiguous. *In mathematics, a statement is acceptable or valid, only if it is either true or false.*

Conjectures

A *conjecture* is a statement which we believe is true based on our mathematical understanding and experience, i.e., our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses.

Take any three consecutive even numbers and add them say, $2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$, $20 + 22 + 24 = 66$. Is there any pattern you can guess in these sums? What is your conjecture about them?

Solution

One conjecture could be

(1) The sum of three consecutive even numbers is even.

(2) Another could be:

The sum of three consecutive even numbers is divisible by 6

Consider Fig. 5.11. The first circle has 1 point on it, the second 2 points, the third has 3 and so on. All possible lines connecting the points are drawn in each case.

The lines divide the circle into mutually exclusive regions (having no common portions). We can count these and tabulate our results as shown:

Number of points	Number of regions
1	1
2	2
3	4
4	8
5	–
6	–
7	–

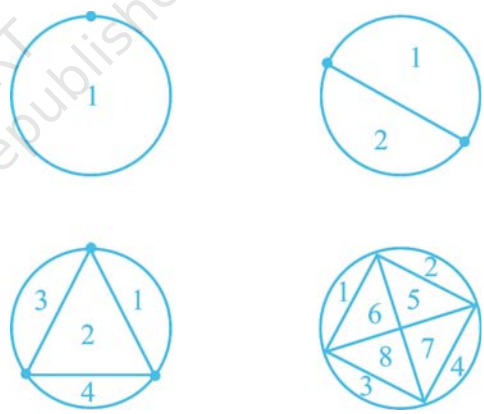


Fig. 5.11

Some of you might have come up with formulae predicting the number of regions, given the number of points. You may recall that such intelligent guess is called a '***conjecture***'.

Suppose your conjecture is "Given 'n' points on a circle, there are 2^{n-1} mutually exclusive regions, created by joining the points with all possible lines". This seems an extreme guess and one can check that if $n = 5$, we do get 16 regions. So, having verified this formula for 5 points, you are satisfied that for any 'n' points, there are 2^{n-1} regions. If so, how would you respond if someone asked you, how can you be sure about this for $n = 25$(say)? To deal with such questions, you would need a proof which shows beyond doubt that this result is true, or a counter example to show that this result fails for some 'n'. Actually, if you are patient and try it out for $n = 6$. You will find that there are 31 regions and for $n = 7$, there are 57 regions. So, $n = 6$ and $n = 7$ are the counter examples to the conjecture above. This also demonstrates the power of a counter example.

Generalisation

Let us take an example of algebraic identities:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

— — — — —

Can you write the other identities for any natural number, 'n' (say), i.e., can you generalise this result for any natural number 'n'?

Let us consider binomial theorem for any positive integer 'n' or for any natural number 'n'.

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$

The proof is obtained by applying the principle of mathematical induction.

Let the given statement be

$$P(n): (a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n b^n$$

Now, for $n = 1$, we have

$$P(1): (a + b) = {}^1C_0\, a + {}^1C_1 b = a + b$$

Thus, P(1) is true.

Suppose P(k) is true for some positive integer k, i.e.,

$$(a + b)^k = {}^kC_0 a^k + {}^kC_1\, a^{k-1} b + {}^kC_2\, a^{k-2} b^2 + \ldots + {}^kC_{k-1} ab^{k-1} + {}^kC_k b^k \qquad (1)$$

We shall see that P (k + 1) is also true, i.e.,

$$(a + b)^{k+1} = {}^{k+1}C_0\, a^{k+1} + {}^{k+1}C_1\, a^k b + {}^{k+1}C_2\, a^{k-1} b^2 + \ldots + {}^{k+1}C_k\, ab^k + {}^{k+1}C_{k+1} b^{k+1}$$

Now,

$$(a + b)^{k+1} = (a + b)\,(a + b)^k$$

$$= (a + b)\,({}^kC_0\, a^k + {}^kC_1\, a^{k-1} b + {}^kC_2\, a^{k-2} b^2 + \ldots + {}^kC_{k-1}\, ab^{k-1} + {}^kC_k\, b^k) \quad [(\text{From}(1))]$$

$$= {}^kC_0\, a^{k+1} + {}^kC_1\, a^k\, b + {}^kC_2\, a^{k-1}\, b^2 + \ldots . + {}^kC_{k-1} a^2\, b^{k-1} + {}^kC_k\, ab^k + {}^kC_0\, a^k b + {}^kC_1 a^{k-1} b^2$$

$$+ {}^kC_2 a^{k-2}\, b^3 + \ldots . + {}^kC_{k-1} + ({}^kC_2 + {}^kC_1)\, {}^{k-1}b^{\,2} + \ldots . + ({}^kC_{k-1} + {}^kC_k)\; ab^k + {}^kC_k b^{k+1}$$

(Grouping)

$$= {}^{k+1}C_0 a^{k+1} + {}^{k+1}C_1 a^k b + {}^{k+1}C_2 a^{k-1} b^2 + \ldots . + {}^{k+1}C_k ab^k + {}^{k+1}C_{k+1} b^{k+1}$$

Thus, it is true for P(k + 1), whenever P(k) is true. Therefore, by the principle of mathematical induction, P(n) is true for every positive integer 'n'.

5.7 Difference between Teaching of Mathematics and Teaching of Science

Mathematics is closely related to science in the sense that it provides precise way of expressing scientific results in mathematical terms. That is why mathematics is called language of science. Mathematics can be said to have two branches namely, pure and applied. Applied mathematics deals with solving problems that arise in other fields of study viz., astronomy, physics, chemistry, computer science, biological sciences, etc. It is difficult to make distinction between pure and applied mathematics. There are many theoretical results which have several important applications. For example, Euler's result of purely theoretical interest concerning complex numbers, i.e., $e^{i\theta} = \cos\theta + i \sin\theta$ has tremendous applications in physical sciences. Similarly, problems in applied mathematics give rise to developments in pure mathematics. The major distinction between science and mathematics lies in the fact that science uses mathematics for explaining its principles and results but not conversely, i.e, sciences are not used to explain and prove mathematical results. Mathematics is a creative discipline, an art, and an expression of the human mind motivated by insight, intuition, and a desire to understand the World in which we live.

Another fascinating aspect to draw the line of distinction between science and mathematics lies in the mode of reasoning. To a large extent, science is based on inductive reasoning whereas mathematics in making is inductive but its finished form is deductive. Scientific theories are based on premises (hypotheses), observations and experimentation whereas basic mathematical entities are undefined terms, defined terms, premises (axioms) and logical reasoning leading to theorems. Scientific results are validated through experiments where as mathematical results are validated by reasoning and logic. Truly speaking, in its background, the notion of science is defined whereas no precise definition could be given to the notion of mathematics.

Keeping in view the above discussion, teaching of mathematics is different from teaching of science in the following ways:

(1) Teaching of mathematics starts with undefined/defined terms, axioms, theorems whereas teaching of science starts with well defined terms.

(2) Mathematics is a man-made Universe while science is a natural one.

(3) In teaching of science, the results are validated through experiments, in teaching of mathematics experimentation are the means to verify some of the probable results.

(4) Teaching of science involves inductive process while teaching of mathematics involves a combination of both inductive and deductive processes.

In support of the above points students can explore the situational examples.

EXERCISE 5.1

1. What is the best method of teaching mathematics in your view? Support your answer by giving reasons and arguments.
2. Which method is commonly used by the present mathematics teachers in teaching mathematics to school children? Write its limitations. How can we improve this method according to present situation?
3. Write and explain the importance of 'concept formation' model of teaching mathematics. Explain its importance by taking one example each from Arithmetic, Algebra and Geometry.
4. What do you understand by 'Problem-solving approach' What is the place of this method in teaching mathematics? How will you use this method in your mathematics classes? Give three important benefits using problem-solving approach to teaching of mathematics.
5. Write stages of problem-solving approach. How is it useful in daily life? Discuss with examples.
6. Differentiate between inductive and deductive methods. In your opinion, which is better – using inductive method and deductive method separately or using them together. Justify your answer with examples.
7. What are the limitations of Deductive method?
8. Mention the four advantages of Deductive method?
9. Discuss the importance of Inductive-Deductive method of teaching mathematics with concrete examples.
10. How can you use the activity based approach in your teaching and what are the advantages of teaching by this method?
11. If you are required to teach all the students of your class collectively, which teaching method will you use and how will you use it?
12. Which teaching method will you use to teach the topic algebraic identities in Class VIII? Explain by examples.
13. Give the merits and demerits of discovery method (Heuristic method) and explain its utility in teaching of mathematics.

14. A practical mathematics teacher cannot make a particular teaching method as the basis of his teaching. The maximum possibility can be that her/his teaching method will be a combination of all the methods. Explain.

15. Simple interest $= \dfrac{\text{Principal} \times \text{Time} \times \text{Rate}}{100}$. Which method will you use for teaching the derivation and use of this formula?

16. Give two reasons to clarify why discovery method is a better method of teaching mathematics as compared to other methods.

17. Discuss the procedure to adopt for developing the understanding of "Ratio and Proportion" to the students of Class VII. Illustrate with concrete examples.

18. Which teaching method will you use to solve the following problem?

    A heap of wheat is in the form of a cone whose diameter is 10.5 m and height is 3 m. Find its volume. The heap is to be covered to protect it from rain. Find the area of the canvas required. Explain different steps of your method.

19. Which method do you prefer to teach 'Remainder Theorem' in Class IX, explain in detail?

20. Which teaching method will you use to solve the following problem?

    In a mixture of 60 litres the ratio of milk and water is 4:1. How much water must be added to make the ratio 3:2? Explore different steps of your method.

Summary

This Unit discusses various methods of teaching mathematics, which enables the teacher to make her/his teaching more interesting and creative. These methods help the teacher to transact the contents of mathematics to the students. These methods are discussed with the help of examples to consider the practical aspects of their utilisation in the real classroom situation. Also the generalisation of mathematical concepts, conjectures and difference between teaching of science and teaching of mathematics have been discussed in this Unit.

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UNIT

6

PLANNING FOR TEACHING - LEARNING MATHEMATICS

6.1 Introduction

Every enterprise – a factory, an institution, a nation, a community needs proper planning for its growth and attainment of the desired goals. In the same way, a proper planning is needed for an effective teaching – learning process. Plans for a teacher involve making tentative decisions regarding expectations for a given course and deciding how these expectations can best be accomplished. Plans are formulated for particular units of four or five sessions for each topic. The development of understanding and of competencies is possible through repeated opportunities to use the competencies in different situations and in a variety of ways. Plans for specific lessons involve objectives, strategies and the activities that can be altered or varied according to the levels of interest of the learner.

Carefully constructed plans by a teacher help him/her in accomplishing the predesigned goals. Planning gives a direction and clarifies the thought. Planning helps teachers to become effective classroom leaders, since well prepared plans facilitate efficient use of available class time and instal confidence.

Learning Objectives

After studying this Unit, the student-teachers will be able to:

- divide syllabus into different Units and further each Unit into series of lessons
- state the characteristics of a Unit
- write a Unit plan

- identify the content categories
- select the content for a lesson
- analyse a concept
- plan a lesson that makes use of all available resources
- adopt an appropriate form of presentation for teaching different kinds of lessons
- use ICT in teaching-learning process.

6.2 Unit Planning

Within the comprehensive scope of school curriculum, the individual teacher needs to chart yearly course. Within this setting, he/she decides which parts of the year's work, if any, should be converted into Units of learning.

Broadly speaking, a Unit is a division of a course. It includes all the materials that cover a specific topic. The division of a course into Units helps to provide a framework for effective learning.

Definition of a Unit Plan: A Unit plan is a comprehensive series of meaningful learning experiences built around a central theme or idea and organised in such a way as to result in appropriate behavioural changes in pupils.

A Unit plan may extend from a minimum of 2-3 days duration to one week or month or so depending on the content.

6.2.1 Characteristics of a Unit Plan

1. The Unit must be unified.
2. It should fit into content.
3. It should be reasonably comprehensive.
4. It should reflect the abilities and interests of the students.
5. It should ensure how to motivate the students.
6. It should include provision of a variety of methods of teaching.
7. It should contain problem-solving activities.
8. It should provide scope for evaluation.
9. It should provide necessary resources.

6.2.2 Unit Plan on Similarity of Figures

Introduction

Other day in the morning at about 9 a.m. a few persons were measuring the shadows of a school building and a lamp post in front of the building. As I enquired, I was told that they were estimating the height of the building by knowing the lengths of the shadows of the building and that of the lamp post and also the height of the lamp post. Thus, to know the height of the building, one need not measure it directly. Often, those which cannot be directly measured, can be measured indirectly. The height of a hill or light house, the distance of a boat from the foot of a light house or sea shore, the altitude of a kite above the ground level, are some such instances. None of these can be measured directly.

The idea that helps one in all such situations is similarity of figures.

Figures which are of the same shape (but not necessarily of the same size) are called similar figures. Similarity is an important concept and a very useful one. A mathematician sees this as an important concept since he is able to recognise order and pattern both in nature and in the mathematical world of abstraction. An architect and an engineer find the idea of similarity useful because of the aesthetic value it adds to the construction, besides the contribution of the idea to the strength and life of the structure. Looking at the structure of petals of flowers, leaves and their spread, a botanist perceives that the idea of similarity is useful in classification. A layman who visits temples and monuments can appreciate the worthwhileness of the idea of similarity as it adds to the beauty and aesthetics of decorative art and sculpture. The relationship between algebraic notions, such as ratio, proportion and similarity helps a mathematician to establish mathematically, results concerning similarity, which remain, otherwise, only an empirical and intuitive idea based on imagination. Such an interaction between algebraic processes and geometrical ideas helps one to apply the results and principles in new situations to seek solutions of geometric problems and vice versa.

In contrast to the traditional treatment of similarity, where emphasis is on memorisation of definitions and results, the stress here is on understanding of the concept and its detail. The Unit deals with the basic concept and the related ones. The approach to the study of similarity is based on notions of Euclidean geometry.

I. Instructional Objectives

The pupil acquires

1. the *knowledge* of similarity – terms, concepts, principles, etc.

Specifications

(a) *Defines* terms, concepts, principles, etc.

(b) *Explains* terms, concepts, principles, etc.

2. **understanding** of similarity – terms, concepts, principles, processes, etc.

Specifications

(a) Illustrates

(b) Compares and Contrasts

(g) Interprets

(h) Classifies

(c) Discriminates between
(d) Finds relationship between
(e) Cites examples of
(f) Constructs examples of
(i) Verifies
(j) Generalises
(k) Analyses
(l) Follows known processes

Proofs – in respect of the terms, concepts, principles, processes, etc.

3. **The ability to apply** the knowledge and understanding of similarity in unfamiliar situations.

Specifications

(a) Locates the problems involving the concept of similarity
(b) Analyses the problems
(c) Checks the adequacy of the data
(d) Selects appropriate method
(e) Estimates the result
(f) Suggests alternate (or new) methods of solving-problems
(g) Solves problems (unfamiliar situations)
(h) Formulates hypotheses from the data
(i) Finds new applications

4. **Constructional and computational skills for the concept of similarity**

Specifications

(a) Uses geometrical instruments appropriately and efficiently.
(b) Works out numerical problems accurately and with reasonable speed.
(c) Detects errors in a procedure.
(d) Makes accurate drawings, measurements, etc.

II. Major Concept

Two figures (or objects) having the same shape (not necessarily the same size) are similar.

III. Content Analysis - Terms and Concepts

(i) Parallel lines
(ii) Concentric circles
(iii) Congruent and similar triangles
(vii) Equiangularity
(viii) Centre of similitude
(ix) Symmetric figures and line of symmetry

(iv) Ratio and Proportion
(v) Ratio of similitude
(vi) One-one correspondence
(x) Idea of a scale
(xi) Perspective
(xii) Projection

IV. Content Development and Organisation

A student is expected to know some of the terms and concepts before studying a Unit. These terms and concepts form the prerequisites for the Unit and therefore, need to be recalled (or revised).

These are : **(A)** *Parallelism*

Concept (a) **Two lines in a plane are parallel if they do not intersect each other** (See Fig. 6.1).

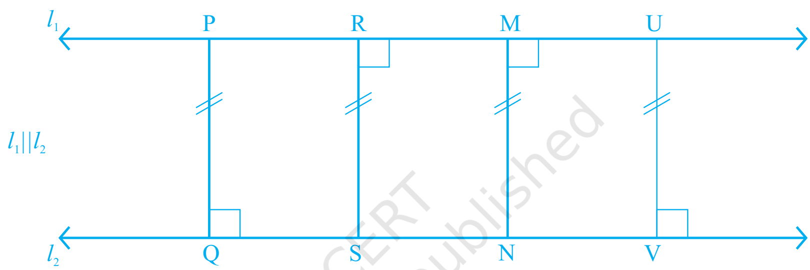


Fig. 6.1

(b) **Two flat surfaces are parallel** if the property in (a) is true here also.

Analogies (or Examples)

(i) The two rails of a railway track are parallel

(ii) The opposite edges of a rectangular door are parallel

(iii) The opposite (vertical) edges of a wall are parallel

(iv) The floor of a house is parallel to the flat roof

(v) Opposite walls of a house are parallel

(vi) In a rectangular box, the opposite edges are parallel and the opposite faces are parallel.

(B) Concentric Circles are circles having the same centre.

Analogies (i) The outer and inner rims of a wheel are concentric circles

(ii) The circular waves spreading out from a point in a pond are concentric circles.

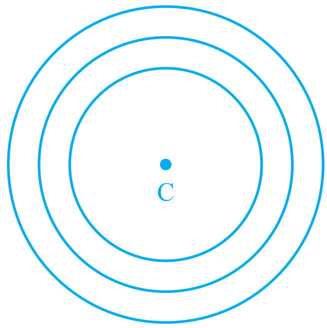


Fig. 6.2

(C) **Congruent Triangles** are triangles in which corresponding sides are equal and hence, their corresponding angles are equal

Two triangles are congruent under the following conditions:

(a) Two sides and the included angle of one are respectively equal to two sides and the included angle of the other (SAS)

(b) Three sides of one are respectively equal to three sides of the other (SSS)

(c) Two angles and the included side of one are respectively equal to two angles and the included side of the other (ASA)

(d) Two right triangles are congruent if the hypotenuse and a side of one are respectively equal to hypotenuse and a side of the other (RHS).

(D) **Ratio and Proportion: Ratio** is the quotient of the measures of two quantities measured in the **same** units. If the measures are a and b units, then the ratio is written as $\frac{a}{b}$ or $a : b$, a and b are respectively called the antecedent and consequent of the ratio $\frac{a}{b}$. If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal, then we write $\frac{a}{b} = \frac{c}{d}$, or $a{:}b{:}{:}c{:}d$ (read as *a is to b* as *c is to d*). Then, a,b,c,d are said to be in proportion.

If $\frac{a}{b} = \frac{b}{c}$, then a,b,c are said to be in continued proportion and b is called the *mean proportional* between a and c given by

$b^2 = ac$ or $b = \sqrt{ac}$

(E) **Division of a Line – Segment in a Given Ratio.**

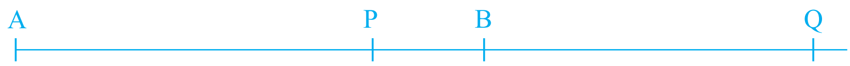


Fig. 6.3

In Fig. 6.3,

(a) If P is such that $\frac{\text{AP}}{\text{PB}} = \frac{a}{b}$, then P is said to divide line segment AB in the ratio $\frac{a}{b}$ *internally*.

(b) If Q is such that $\frac{\text{AQ}}{\text{QB}} = \frac{a}{b}$, Q is said to divide line segment AB in the ratio $\frac{a}{b}$ *externally.*

Obviously, AB *cannot be* divided in the ratio 1 : 1 externally.

Development of New Concepts

Two figures having the same shape are said to be *similar.*

Any two *circles* are similar [Fig. 6.4].

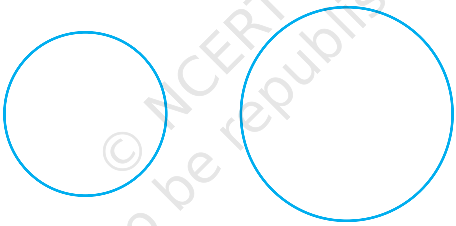


Fig. 6.4

Any two *squares* are similar [Fig. 6.5].

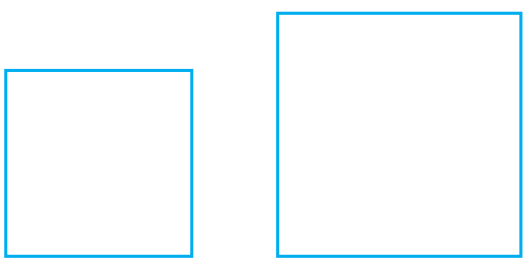


Fig. 6.5

Any two *equilateral triangles* are similar [Fig. 6.6].

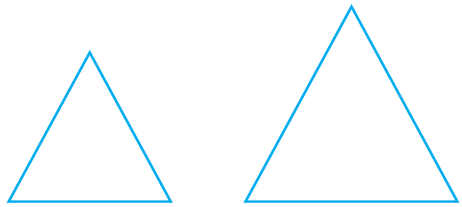


Fig. 6.6

Any two *line segments* are similar [Fig. 6.7].

Fig. 6.7

Analogies and Examples

(a) Two different sized photographs of the same object

(b) Any two maps of a country, drawn on different scales

(c) A picture and its projected image on a screen

1. Similarity of Polygons

(a) **Conditions for similarity**

(i) Corresponding angles must be equal.

(ii) Corresponding sides must be proportional.

(b) In the case of similar triangles, any one of the two conditions mentioned above is sufficient.

2. One-one Correspondence

When two similar triangles are considered, the pair of sides opposite to equal angles are described as *corresponding sides*. In turn, the pair of equal angles are *corresponding angles*. These ideas are expressed as

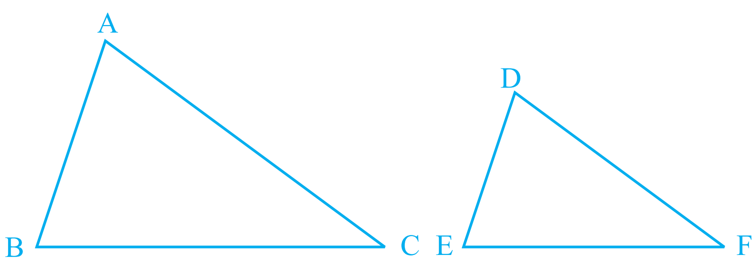


Fig. 6.8

$$\begin{array}{ccc} \overline{AB} & \leftrightarrow & \overline{DE} \\ \overline{AC} & \leftrightarrow & \overline{DF} \\ \overline{BC} & \leftrightarrow & \overline{EF} \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{A} & \leftrightarrow & \hat{D} \\ \hat{B} & \leftrightarrow & \hat{E} \\ \hat{C} & \leftrightarrow & \hat{F} \end{array}$$

The way in which the sides (or angles) of one triangle are associated with the sides (or angles) of the other in a one-one manner is called a *one-one correspondence*.

More generally, in case of two similar polygons ABCDEF... and PQRSTU... with pairs of sides *a,p; b,q; ,c,r*; etc. the one-one correspondence of sides is given by

$a \leftrightarrow p$; $b \leftrightarrow q$; $c \leftrightarrow r$; $d \leftrightarrow s$; etc.

3. Equiangularity

This concept is implicit in the foregoing discussions and definition of similarity. Two similar figures (rectilinear) must necessarily be equiangular. In the case of similar triangles, the condition of equiangularity (the fact that the angles of one are equal to those of the other) is necessary as well as sufficient.

4. Centre of similitude (or Similarity) or Homothetic Centre

Whenever similar rectilinear figures have their corresponding sides parallel, the lines joining the corresponding vertices of similar rectilinear figures are concurrent at a point called the centre of similarity of the figures.

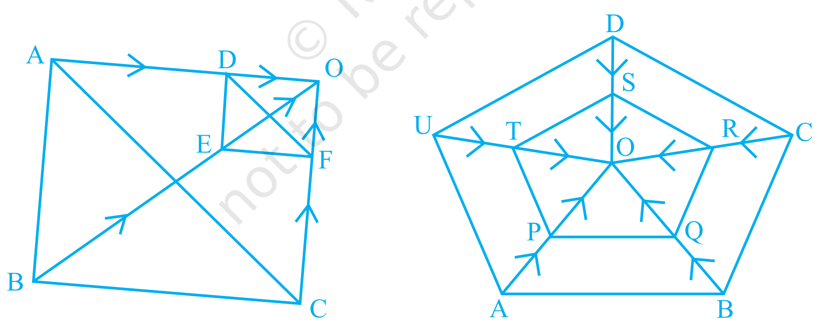


Fig. 6.9

O is the centre of similarity in each of the above figures.

5. Perspective: Whenever similar figures are such that the lines joining their corresponding vertices meet at a point (the centre of similarity), the figures are said to be in *perspective*. For two figures to be in perspective, the centre of similarity must exist.

6. Idea of Scale: You know that, for a map of a country or a sketch of a figure to be a true representative, it must be similar to the original. This happens when the ratio of distance on the map to the actual distance is constant. This ratio of distances is called the *scale*.

If a distance of 40 metres is represented by a length of 5cm, then the scale is $\frac{5}{40 \times 100} = \frac{1}{800}$ which is called the *representative fraction.*

7. Projection

When an object is in front of a source of light, the shadow of the object is called the *projection* of the object [Fig. 6.10].

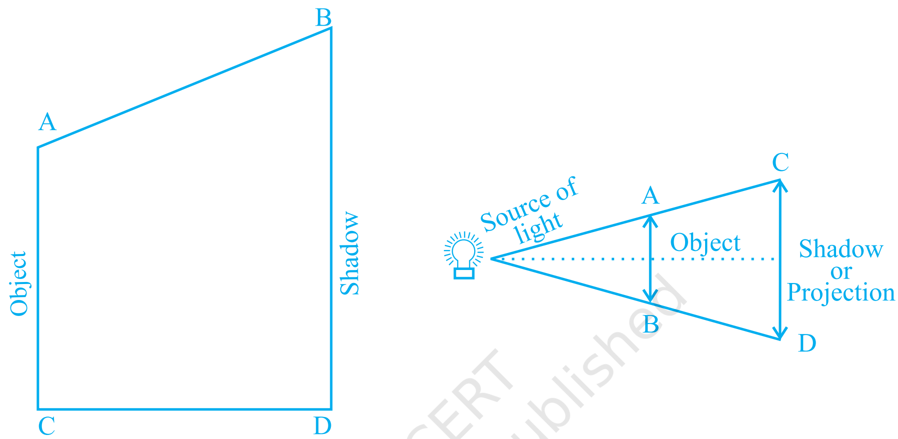


Fig. 6.10

8. When an object is held before a plane mirror, the object and the image or reflection of the object are congruent and hence similar object. The line representing the mirror is the *line of symmetry* as the object and the reflection are symmetrical (see Fig. 6.11).

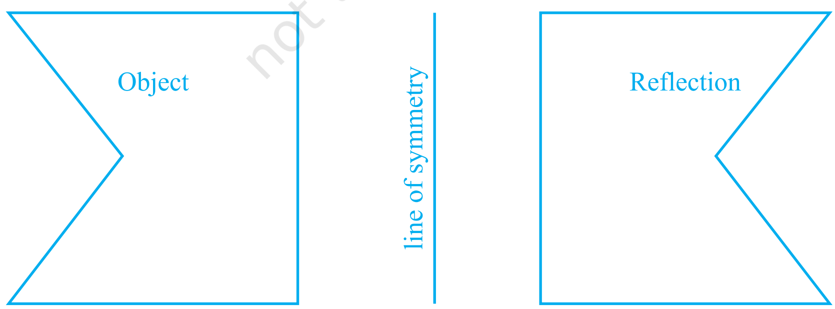


Fig. 6.11

V. Learning Experiences (Activities and Techniques)

1. Recalling a number of analogies and examples of objects and situations involving the idea of similarity.

2. Explaining the usefulness of the idea of similarity and asking the pupils to narrate or list some more uses of similarity.
3. Asking them to reason out their inferences.
4. Giving practice in drawing similar figures and encouraging them to make models of similar figures and objects.
5. Letting them identify one-one correspondence between various components of similar figures.
6. Asking them questions regarding
   (a) Conditions under which two figures are similar.
   (b) Conditions when similar figures are congruent
   (c) Construction of similar figures.
7. Letting them construct similar figures.
8. Asking them to determine whether the figures are in perspective.
9. Asking them to reason out why two given figures are *not* similar.
10. Encouraging them to know more about similarity and the applications of the idea.
11. Helping them to construct alternate proofs to known results.
12. Helping them to solve (a) new riders (b) new problems and (c) problems in new and practical situations, involving the notion of similarity.
13. Asking them to classify objects and figures on the basis of similarity and also examine similar figures to know their related properties.
14. Encouraging them to construct and improvise models of objects/figures which are similar and which display the properties of similar figures.
15. Providing them challenges by way of tests, puzzles, problems, etc., involving the idea of similarity.

Some of the activities

(a) can be spread over a period of time
(b) can be conducted both inside and outside the classroom
(c) can be conducted as group activity for various groups.

Teaching Hints: The concept of similarity may be introduced in a variety of ways — through (a) analogies, (b) examples, (c) situations, (d) problems, (e) practical problems and (f) the pupil's aesthetic sense etc. The advantages of such an introduction are many. One important advantage is that it provides the necessary motivation. The pupils will know how far the idea is worthwhile, useful and from how many angles, the concept can be studied. All this gives them a sense of satisfaction and the healthy impression that whatever they are learning is meaningful and worth.

As far as the methods and techniques are concerned, emphasis must be on learning by doing and not teaching by talking or teaching by demonstration by the teacher himself/ herself.

Evaluation at convenient intervals, at the end of units/sub units is very useful. It builds confidence in students, informs the less capable their deficiencies and provides feedback to the teacher who can alter his/her techniques.

VI. Theorems and Results

1. A line segment can be divided in a given ratio internally/ externally at one and only one point.
2. If a pair of transversals intersect a set of parallel lines, the intercepts made on the pair are proportional [Fig. 6.12]

i.e., $\frac{PQ}{P'Q'} = \frac{QR}{Q'R'} = \frac{RS}{R'S'} = L$

or, $PQ : QR : RS$

$= P'Q' : Q'R' : R'S'$

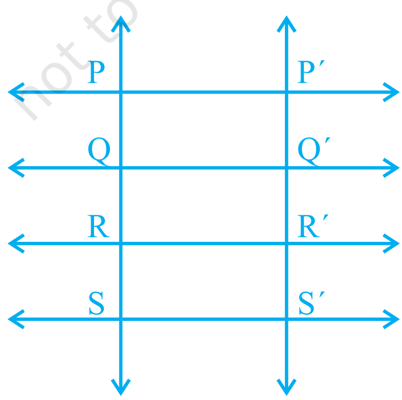


Fig. 6.12

3. A line parallel to a side of a triangle divides the other sides in the same ratio.

i.e., $\frac{AX}{XB} = \frac{AY}{YC}$ [Fig. 6.13]

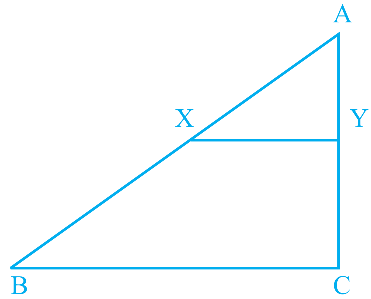


Fig. 6.13

4. The internal or external bisector of an angle of a triangle divides the opposite side internally or externally in the ratio of the other sides of the triangle [Fig. 6.14].

i.e. , $\frac{AB}{AC} = \frac{BP}{PC}$ and $\frac{AB}{AC} = \frac{BQ}{QC}$

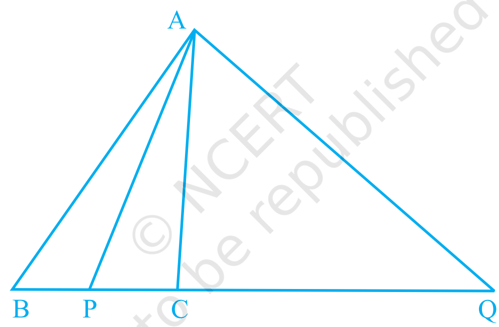


Fig. 6.14

5. Two triangles are similar (i) if they are equiangular, or (ii) if the sides of one are proportional to those of the other, or (iii) if an angle of one equals an angle of the other and the sides including these equal angles are proportional.

6. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the two triangles obtained are similar and each is similar to the given triangle.

7. The areas of similar triangles are proportional to the squares of the corresponding sides.

8. The perimeters of similar triangles are proportional to the corresponding sides of the triangles.

9. The perimeters of similar polygons are proportional to the corresponding sides and the areas of the similar figures are proportional to the squares on the corresponding sides.

10. Similar polygons with parallel corresponding sides are in perspective, i.e., the lines joining corresponding vertices are concurrent (at the point called the *centre of similarity*).
11. If similar polygons or semicircular figures are constructed on the sides of a right triangle, then the area of the figure on the hypotenuse equals the sum of the areas of the figures on the other sides.
12. If the bisectors of the base angles of a triangle meet the opposite sides at X and Y respectively and if XY is parallel to the base, then triangle is isosceles.
13. In Fig. 6.15, AD is a median of ΔABC and the bisectors of the angles ADB and ADC meet AB and AC, respectively at E and F, then EF is parallel to BC.

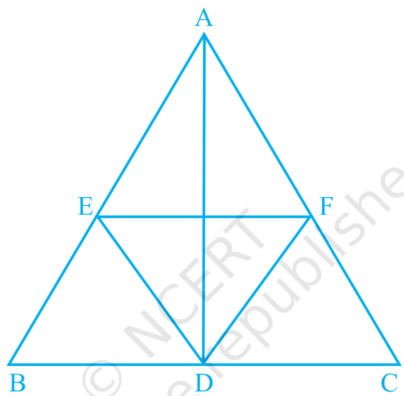


Fig. 6.15

14. If bisectors of the angles of A, B and C of ΔABC meet the sides BC, CA and AB at D,E,F respectively, then BD.CE.AF = DC.EA.FB.
15. In Fig 6.16, BC = DC and the bisectors of the angles ACB and ACD meet AB and AD at E and F respectively. Then EF || BD.

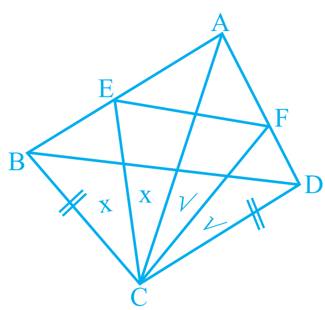


Fig. 6.16

16. If the sides of a triangle are parallel to the sides of another triangle respectively, then the triangles are similar (see Fig. 6.17.)

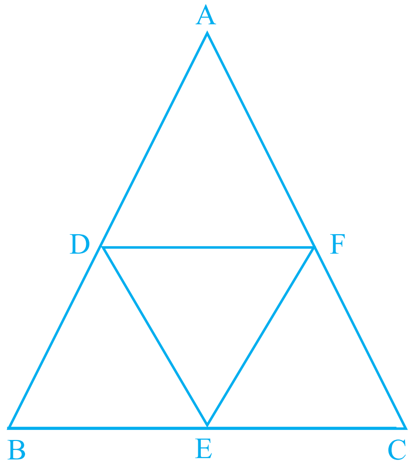


Fig. 6.17

VII. Evaluation (Sample Test Items)

1. Which condition(s) for similarity is (are) not satisfied for the following pairs of figures?

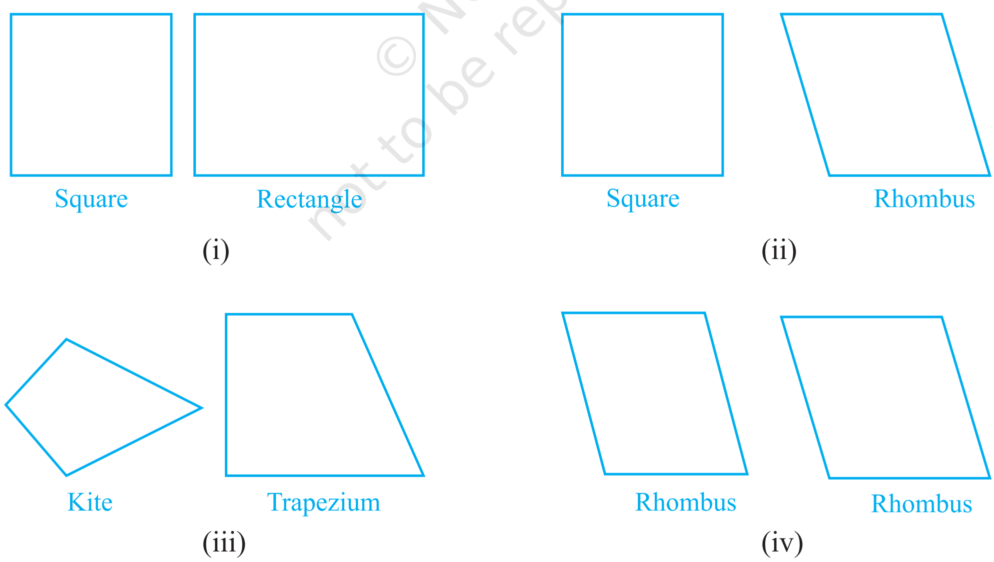


Fig. 6.18

2. Which of the following classes of figures are of similar figures?

(i) Squares (ii) Parallelograms

(iii) Equilateral triangles (iv) Isosceles triangles

(v) Rectangles (vi) Circles

(vii) Rhombuses (viii) Kites

3. Given a triangle, is it possible to divide it into a number of similar triangles? If so, how do you do it ?

4. P divides AB = 10cm in the ratio 5:2. Find AP if P divides AB (a) internally (b) externally.

5. In the triangle ABC, AB = 6cm, AC = 4cm. If AE divides BC in the ratio 3:2, then which of the statements is true? [Fig 6.19]

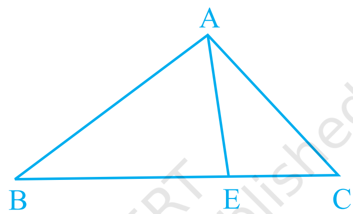


Fig. 6.19

(i) AE is a median

(ii) AE is an altitude

(iii) AE is an angle bisector of the angle BAC.

6. In Fig. 6.20, how many sets of similar triangles can you search ?

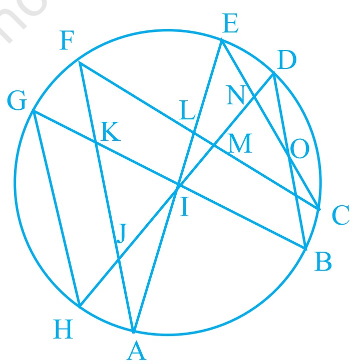


Fig. 6.20

7. For Fig. 6.21, which of the following statements is/are true?

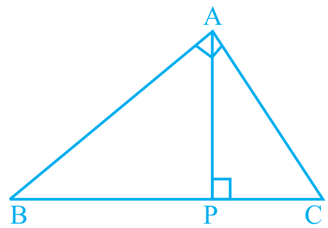


Fig. 6.21

(i) PA.PB = PC^2

(ii) PB.PC = PA^2

(iii) PA.PC = PB^2

8. Which statement is true?.

(i) When the vertices of similar figures are joined in pairs, the lines joining them are concurrent.

(ii) When the corresponding vertices of similar figures are joined, the lines joining them are concurrent.

(iii) When the corresponding vertices of similar figures with corresponding sides parallel, are joined, the lines joining them are concurrent.

9. Describe *five* situations in life, where similarity idea comes into play.

EXERCISE 6.1

1. Explain the purpose of 'introduction' in unit planning.
2. What is the role of 'content analysis' in unit planning?.
3. How does a unit plan differ from the presentation and development of subject matter in a textbook?.
4. Discuss the role of evaluation in unit planning. Prepare a unit plan on one unit each in (i) Arithmetic (ii) Algebra (iii) Geometry.

6.3 Lesson Plan

6.3.1 Need for Planning

Educators have always agreed on the need for an intelligent planning of every lesson. A plan is a blue print which helps in an efficient, economical and smooth conduct of any activity. If teaching is to be effective in terms of learning by students, it is necessary to ensure this through careful advance planning which would involve visualising the entire teaching-learning situation as it is likely to develop in the classroom. Every teacher has before him/her some very specific purposes in teaching a topic of a Unit. He/she is conscious to achieve these purposes during the course of the lesson. He/she needs to think about the best possible manner in which he can realise these purposes with maximum efficiency and the minimum wastage of available resources. This applies to all teachers, the need for such advance planning is all the greater for the beginners. This is because of the newness of teaching, subject matter, students and the school. Careful planning helps develop the teacher's mathematical competence that enhances his/her confidence in front of the students, and eventually provides the context for developing creative talent in teaching.

6.3.2 Advantages of Planning

(1) Planning clarifies thought.

(2) Planning infuses confidence.

(3) Planning gives a direction and makes smooth sailing through out entire period

(4) Planning makes the teacher to show himself to the students that a teacher is well organised, authority figure in teaching-learning.

(5) Planning includes the selection of the content, structuring the content, sequencing the content so as to make the lesson easy to follow and easy to understand.

(6) It is helpful in making the students remember and identify the important things that are worth remembering and develop reasoning abilities among them.

Now we consider some basic elements in lesson planning.

6.3.3 Content Categories in Mathematics

Content in mathematics exists in three primary forms : *facts, concepts and generalisations.*

Fact : (Singular statements): Fact is defined as the type of content which is singular in occurrance, which has occurred in the past or exists in the present, but has no predictive value. Facts are acquired solely through the process of observation.

Example: (1) π is an irrational number

(2) The base of the common logarithmic function is 10.

Concept: Concept is defined as the type of content which results from the categorisation of a number of observations.

Or

A decision rule which, when applied to the description of an object, specifies whether or not a name can be applied.

Example: (1) Triangle

(2) Prime number

Generalisation: It is an inferential statement which expresses a relation between two or more concepts, applies to more than one event and has predictive and explanatory value.

Example: (1) The diagonals of a rectangle bisect each other.

(2) The slope of a linear function is constant.

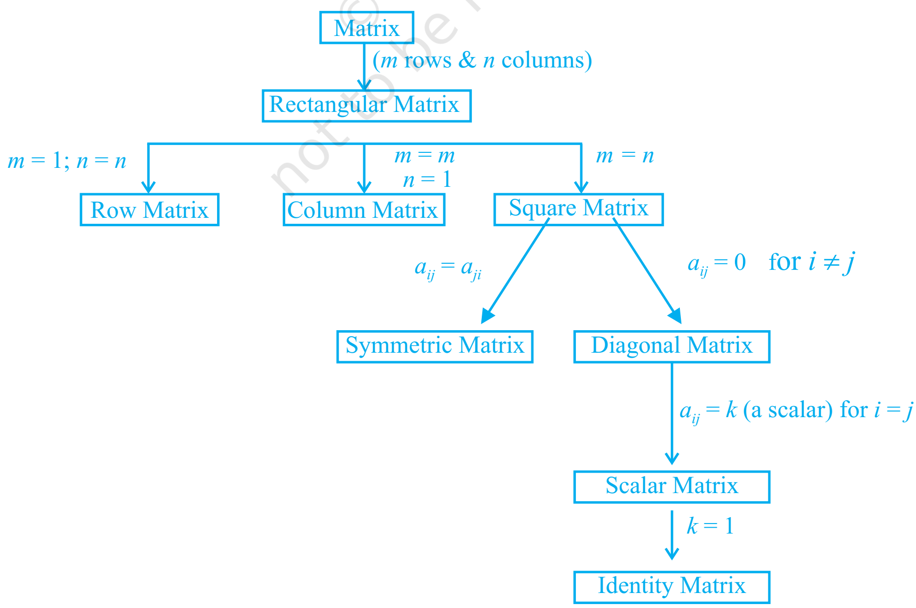
6.3.4 Selection of Content

Selection of the content for a lesson involves a number of considerations. The selection of the content should be related to the overall programme of study for pupils. The first and foremost task of the teacher is to separate a topic into distinct elements or aspects like concepts, generalisations, facts and prescriptions and to design a sequence or progression in a hierarchical way through which these elements that make coherent and intellectual sense, effectively and facilitate learning and retention. He has to discriminate between essential and unimportant matters within the subject.

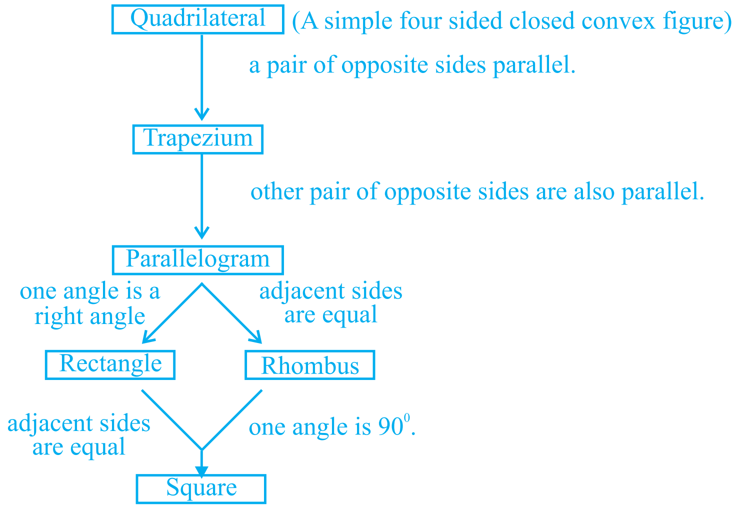
6.3.5 Concept Analysis

Teaching concepts in isolation, without building relationship among the concepts, do not develop the processing skills, that are transferable to real World problems. Teachers should be aware of the relationship between concepts, i.e., concept mapping and also to analyse a concept. Concept maps represents a collection of interconnected concepts with specified relationships between pairs of concepts and the links connecting them. Concept maps are hierarchical in nature.

Concept Map of the Concept "Matrix"



Concept Map of the Concept "Quadrilateral"



Analysing a concept by a teacher helps him/her to give clear definitions, more and appropriate examples and non-examples. Concept-based instruction links prior instruction to new instruction.

Example 1

Concept Name : Perpendicular Bisector

Concept Definition : Perpendicular Bisector of a given line segment is a perpendicular line that divides the given line segment into two equal parts.

Essential Attributes

1. The line intersects the given line segment at right angles.
2. The line divides the given line segment into two equal parts.

Non-Essential Attributes

1. The length of the line segment.
2. Configuration of the line segment.

Examples:

1.

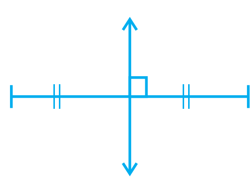


Fig. 6.22

2.

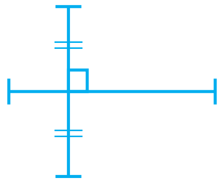


Fig. 6.23

3.

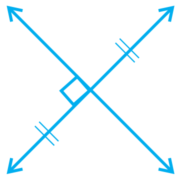


Fig. 6.24

4. Alphabet T
5. Vertical bar in a two pane window.

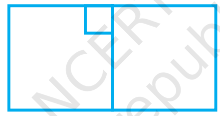


Fig. 6.25

6. Simple balance at rest.

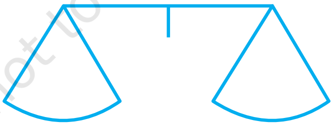


Fig. 6.26

7. CD is perpendicular bisector of AB in an isosceles triangle in the Fig. 6.27.

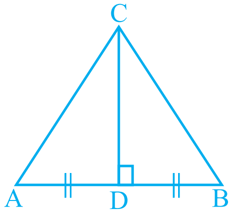


Fig. 6.27

8. Perpendicular bisector of a chord passes through the centre of the circle.

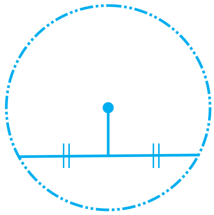


Fig. 6.28

Non-Examples

1.

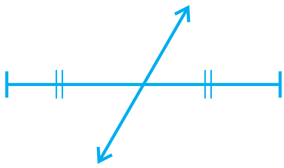


Fig. 6.29

2.

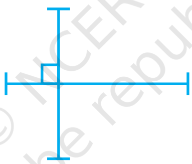


Fig. 6.30

3.

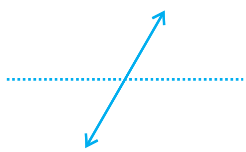


Fig. 6.31

4. Vertical post of a football goal.

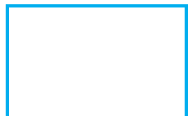


Fig. 6.32

5. Vertical post in a door.

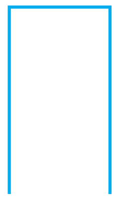


Fig. 6.33

6. Radius of a circle drawn through the point of contact of a tangent is perpendicular to the tangent.

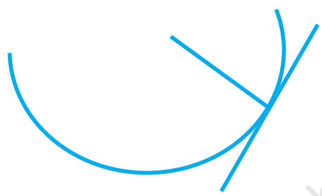


Fig. 6.34

Superordinate Concept : Line

Co-ordinate Concept : Transversal

Note: Perpendicular bisectors shown in the figures are line-segments contained in the respective lines.

Example 2

Concept Name : Identity Matrix

Concept Definition

Identity matrix is a square matrix with principal diagonal elements unity (1) and all other elements zero(0).

Essential Attributes

(i) It is a square matrix.

(ii) Each of the principal diagonal elements is equal to 1.

(iii) Each non-principal diagonal elements is equal to zero.

Non-essential Attributes

Order of the square matrix.

Examples

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Non-examples

(i) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is not a square matrix (A matrix but not an identity matrix).

(ii) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ has non-principal diagonal elements are not zero (A square matrix but not identity matrix).

(iii) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ Principal diagonal elements are not unity (A diagonal matrix but not an identity matrix).

(iv) (a) $\begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$ Principal diagonal elements are not unity and all other elements are non-zero.

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Principal diagonal elements are not unity and non-diagonal elements are non-zero.

Super-ordinate concept : Square matrix

Example 3

Concept Name : Latus Rectum of a parabola

Concept Definition : A *latus rectum* of a parabola is a *line segment* which is perpendicular to the axis of the parabola that passes through the focus and whose end points lie on the parabola.

Essential Attributes

(i) It passes through the focus.

(ii) It is perpendicular to the axis of the parabola.

(iii) Its end points lie on the parabola.

Non-Essential Attributes: Orientation and opening of the parabola.

Examples

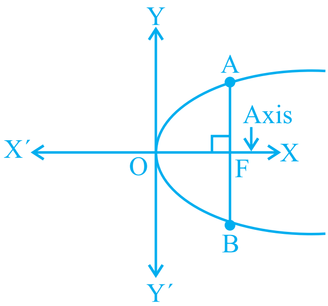


Fig. 6.35

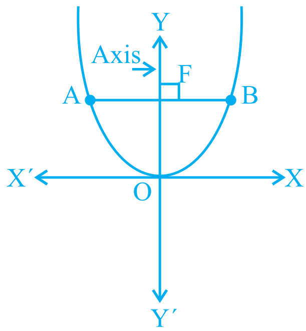


Fig. 6.36

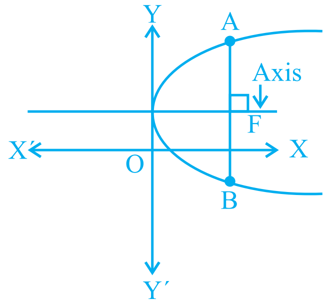


Fig. 6.37

Non-Examples

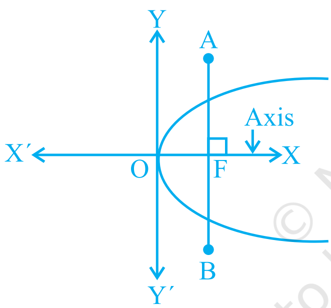


Fig. 6.38

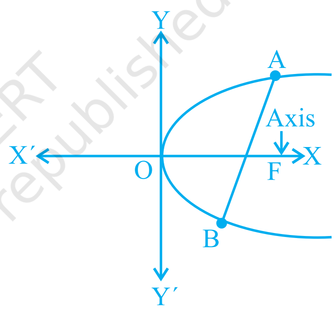


Fig. 6.39

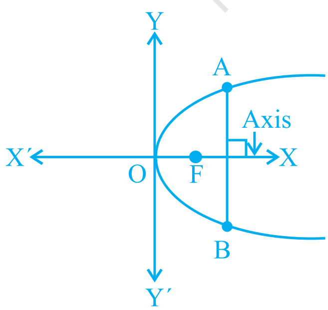


Fig. 6.40

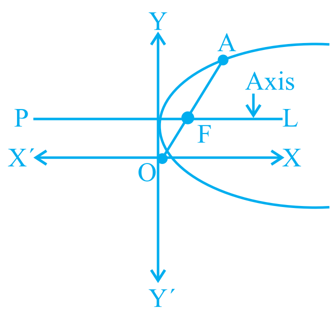


Fig. 6.41

Super-Ordinate Concept : Line Segment.

Example 4

Concept Name : Symmetric Matrix

Concept Definition : A *symmetric matrix* is a square matrix such that the *(i, j)th element is equal to (j, i)th element, for every i and j.*

Essential Attributes : (i, j)th element must be equal to (j, i)th element.

Non-Essential Attributes : Order of the square matrix.

Examples

1. $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ 1 & -3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}$

Non-Examples : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & 1 & 0 \end{bmatrix}$

Superordinate concept: Square Matrix

Example 5

Concept Name : Equivalence Relation

Concept Definition : An *Equivalence Relation* on a set is a *binary relation* which is reflexive, symmetric and transitive.

Essential Attributes : Reflexive property, symmetric property and transitive property.

Non-Essential Attributes : Type of set is non-essential and type of relation is non-essential.

Examples

The binary relation

1. "is equal to" (=) in Q or R is an equivalence relation.
2. "is congruent to" in the set of all triangles.
3. "is similar to" in the set of all triangles.
4. "has same area as" in the set of all triangles.
5. "has same perimeter as" in the set of all triangles.
6. "is parallel to" in the set of all lines in a plane.

Non-Examples

A binary relation

1. "is a subset of" in the set of all sets.
2. "is less than" in the set of all rational numbers.
3. "is a factor of" in the set of all natural numbers.
4. "is perpendicular to" in the set of all lines in a plane.
5. " is greater than" in the set of all integers.

Conceptual Hierarchy

1. **Super-ordinate Concepts** – Binary Relation.
2. **Coordinate Concepts** – Anti-symmetric relation, inverse relation.

Evalution Items: Write the concept analysis of the following mathematical objects:

(a) median (b) adjacent angles (c) function
(d) power set (e) singular matrix (f) right angled triangle
(g) linear pair of angles

6.3.6 Elements of a Lesson Plan

There is no unique way of preparing a lesson plan. One could think of a variety of formats to suit different situations. It is not very important that a particular form for drawing up a lesson plan should be ritualistically followed. What is important is to recognise that a distinctive purpose of a lesson plan is good teaching that has to follow it. A plan must help to clarify to the teacher the specific learning outcomes in pupils, in relation to the topic and indicate how these are proposed to be realized and evaluated.

While there need be no rigidity about the format or pattern of a lesson plan, it may be suggested that the following essential elements find a place in every good lesson plan.

1. Statement of instructional objectives and learning outcomes in relation to the topic. It is desirable to state the objectives in terms of changes in pupil behaviours so that the evidences for the changes might also be sought in pupil behaviours.
2. Selection and sequential organisation of learning activities in terms of the objectives.
3. Selection of appropriate aids and materials and the resources to be used.
4. Selection of appropriate strategies to evaluate the learning outcomes at different stages.

These are explained as given below:

(a) Instructional Objectives

Cognitive Objectives:

(i) Knowledge (Remembering)

An objective requiring students to remember/recall a specified response, but not a multistep sequence of responses to a specified stimulus is at the remembering level.

Knowledge of symbols, terms, definitions, relationships, formulae, conventions, axioms, theorems, etc.

Examples

(1) state the definition of a triangle

(2) state the Pythagoras theorem

(3) state the formula for finding the area of a rectangle

(4) recognise the symbol for summation

(5) state the angle sum property of a triangle

(6) state the SAS axiom for congruence of two triangles

(7) state the laws of surds.

(ii) Understanding (Comprehension)

An objective requiring students to remember a sequence of steps in a procedure and then be able to apply to arrive at a response to a specified stimuli, i.e., this category includes the recognition or recall of specific facts and terminology and the ability to perform a given algorithm in a familiar context. It also includes the ability to translate a verbal description to a pictorial representation, the ability to read and interpret problems, discriminate, compare and contrast, detect errors, give illustrations, etc.

Examples

(1) give examples of an inverse proportion

(2) cite examples from the environment which are in the shape of a square

(3) give the dimensions of a rectangle and compute its area

(4) give reasons for the assertions in the proof of a theorem

(5) explain how the formula for the area of a right triangle can be derived from the formula for the area of rectangle.

(6) classify like and unlike surds

(7) identify surds from a given set of irrational numbers

(8) rationalise the denominator of a given surd

(9) compute the product of two given matrices which are conformable for multiplication.

At the understanding level objectives, one needs to know how to execute the steps in arriving at the response to the stimulus. He recalls the procedure and applies it over.

For Example

If the stimulus is 2+3 for a secondary school student, then the response from the student is immediate and it is a recall from the addition facts rather than figuring it out.

Hence, this is at knowledge level whereas if the stimulus is 86+57, then it requires the student to recall the procedure of addition of two addends, and in place of addends he has to substitute 86 and 57 and the computational procedure is to be carried out to get a response.

Hence, it is an understanding level objective.

(iii) Applying

An applying objective requires students to use deductive reasoning to decide how to utilise, if at all, a particular concept relationship or process to solve problems (Kelley, 1988, pp.169-254). Here "solving problems" is used broadly referring to situations in which students determine strategies for addressing questions and tasks. Solving problems does not include solving exercises which are very routine and no novel thinking is required. It also includes gathering and analysing information, separating a problem into its constituent parts, identifying relevant information for solving a problem, recognising analogous problems, recognising patterns, relationships, generating new patterns or relations, forming hypotheses, infering, predicting and doing other things in new situation. This is the highest level of cognitive behaviour that suggests new or alternate or modified method of attack or a method of solving problems.

Examples: (1) Given the expression $\sqrt[n]{\frac{a}{b}}$, where n, a and b are positive integers, the student should be able to determine whether the given expression is irrational.

(2) When confronted with a real life problem, determine whether or not computing the surface area will help in solving that problem.

(3) Given a polynomial equation and a root of equation, the student should be able to show whether the root is rational or irrational.

(b) Pre-Requisite Knowledge

One needs knowledge to learn. It is not possible to absorb new knowledge without having some structure developed from previous knowledge. New knowledge cannot be learnt in isolation. So, in a classroom, the teacher has to look at the past learning experiences of the students and bring them forward to facilitate present learning, so that a higher degree of learning can be achieved with much ease and takes less time to acquire the new learning. In the context of the lesson plan, the pre-requisite knowledge is that knowledge which the student is expected to possess that will facilitate to learn the new knowledge to be imparted through the lesson. Pre-requisite knowledge does not include all the knowledge he possesses before learning a particular concept or a generalisation. It includes only that knowledge which is required and which is relevant to the present situation.

By reviewing the pre-requisite knowledge at the start of the lesson, pupils can refresh their memories on what they already know. They gain insight into what they are about to learn.

For Example: To learn the proof of the basic proportionality theorem, the pre-requisite knowledge required by the student is

(1) Formula for the area of a triangle, i.e., $\frac{1}{2} \times$ base $\times$ height.

(2) To draw an altitude of an acute angled triangle.

(3) Ability to recognise the two triangles having the same base and lying between the two parallels in different situations and recall that they have the same area.

Note that, it is not necessary that he should know the derivation of the formula $\frac{1}{2} \times$ base $\times$ height, or proof of the theorem "the triangles having the same base lying between the same parallels are equal in area".

(c) Instructional Aids

Now a days, there are a wide range of resources and materials available for use as instructional aid in the classroom like videotapes, slides, mathematical models, overhead projector, transparencies, worksheets, computer software, etc. It is important for a teacher to familiarise himself/herself with the content of such material and plan to use in a classroom situation.

As illustrated below, at times a teacher may also have to prepare himself/herself an instructional aid that will facilitate a student in learning.

Example 1

Objective: To obtain the formula for the surface area of a sphere.

Material Required: A ball, cardboard/wooden strips, thick sheet of paper, ruler, cutter, string, measuring tape, adhesive.

Method of Construction

1. Take a spherical ball and find its diameter by placing it between two vertical boards (or wooden strips) [see Fig. 6.42]. Denote the diameter as *d*.
2. Mark the topmost part of the ball and fix a pin [see Fig. 6.43].
3. Taking support of the pin, wrap the ball (spirally) with string completely, so that on the ball no space is left uncovered [see Fig. 6.43].
4. Mark the starting and finishing points on the string. Unwind the string from the surface of the ball.

   Measure the length between the starting and finishing points on the string and denote it by *l*.
5. On the thick sheet of paper, draw 4 circles of radius '*r*' (which is equal to the radius of the ball).

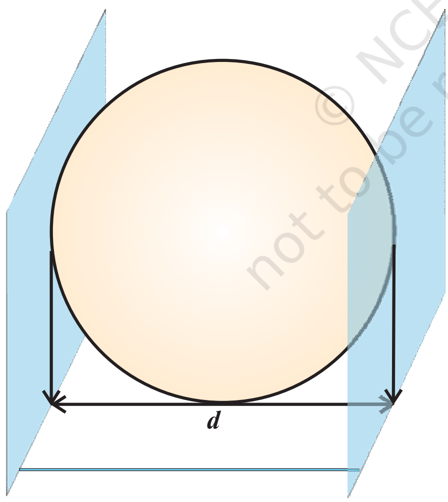


Fig. 6.42

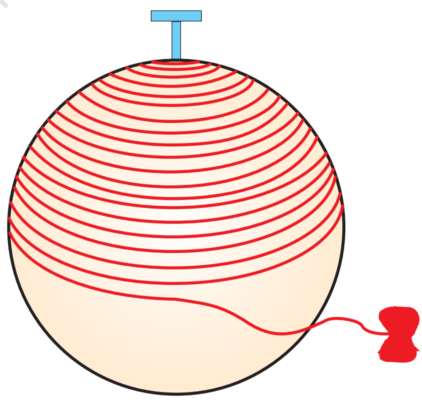


Fig. 6.43

6. Start filling the circles [see Fig. 6.44] one by one with string that you have wound around the ball.

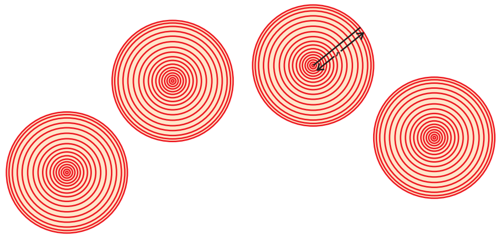


Fig. 6.44

Demonstration

Let the length of the string which covers a circle (radius r) be denoted by a.

The string which had completely covered the surface area of ball has been used completely to fill the region of four circles (all of the same radius as of ball or sphere).

This suggests:

Length of string needed to cover sphere of radius r = 4 × length of string needed to cover one circle

i.e., $l = 4a$

or, surface area of the sphere = 4 × area of a circle of radius r

So, surface area of a sphere = $4\pi r^2$

Example 2

Objective: To find experimental probability of unit's digits of telephone numbers listed on a page selected at random of a telephone directory.

Material Required: Telephone directory, note book, pen, ruler.

Method of Construction

1. Take a telephone directory and select a page at random.
2. Count the number of telephone numbers on the selected page. Let it be 'N'.
3. Unit place of a telephone number can be occupied by any one of the digits 0, 1, ..., 9.
4. Prepare a frequency distribution table for the digits, at unit's place using tally marks.
5. Write the frequency of each of the digits 0, 1, 2, ...8, 9 from the table.
6. Find the probability of each digit using the formula for experimental probability.

Demonstration

1. A frequency distribution table (using tally marks) for digits 0, 1, ..., 8, 9 has been prepared as shown below:

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9

2. Noting down frequency of each digit (0, 1, 2, 3,...,9) from the table, digits 0, 1, 2, 3, ..., 9 are occuring respectively n_0, n_1, n_2, n_3, ..., n_9 times.
3. Calculate probability of each digit, considering it as an event 'E' using the formula

$$P(E)=\frac{\text{Number of trials in which the event occurred}}{\text{Total number of trials}}$$

4. Therefore, respective experimental probability of occurrence of 0, 1, 2, ..., 9 is given by

$$P(0)=\frac{n_0}{N},\ P(1)=\frac{n_1}{N},\ P(2)=\frac{n_2}{N},\ldots,\ P(9)=\frac{n_9}{N}.$$

(d) Learning Experiences

Learning experiences are the activities to be undertaken by the students which are planned deliberately by the teacher with a particular purpose to bring about desirable changes in their behaviour. The selection of learning activities offers much scope and choice for teachers. The decision about which activity or combination of activities to use within a lesson depends on the teachers beliefs about the relative effectiveness of the different activities for the type of learning intended.

Criteria for framing good learning experiences:

1. Learning experiences should be based on the specific objectives to be attained.
2. Learning experiences should be framed in relation to the prescribed syllabus.
3. Learning experiences should meet the needs of the particular age-group of pupils.
4. Learning experiences should provoke good deal of sustained interest in the pupil so as to ensure full participation of the pupil in the experience.
5. Learning experiences should be prepared by taking into account of pupils' abilities, interests and motivation.
6. Learning experiences should provoke the desired reaction in the students.
7. Learning experiences should be closely related to the local environment of the students.

Planning Learning Experiences

When thinking about learning activities to be used, teacher also needs to think of the lesson as a coherent whole, so that the total package of experiences provided for pupils, achieves teachers intended learning outcomes. Thus, not only must the activities promote the appropriate intellectual experience for this learning to occur, but they must also enable pupils to readily engage and remain engaged in this experience. The degree of teacher success will depend on how best the teacher has planned the experiences. So, it is very essential that the teacher knows how to plan learning experiences.

The following points should always be borne in mind while planning the learning experiences:

1. Learning experiences should:
   - be arranged from simple to complex
   - provide the conditions so that conceptualisation occurs first, followed by knowledge and comprehension and finally culminating into application level of learning
   - be organised in such a way that they provide the desired reaction
   - provide and reinforce most of the desired behaviour
   - seek pupils participation to the maximum possible extent
   - be directly linked with the desired goals
   - be arranged according to the facilities available in the immediate environment of the school

   (Students should be engaged in learning activities for conceptualising a concept or relationship. Then, the name for a concept or relationship should be introduced. Before conceptualising the concept or relationship, memorising words to attach a concept or relationship is meaningless for most of the students.)

2. The initial phase of the lesson requires the activities to be designed to set the scene and elicit interest and introduce the topic. The second phase which is the major part of the lesson planning should involve main learning experiences that engage the pupils thinking process and holds their attention, while he/she asks questions, discusses a new skill or concept. The final phase in which the teacher assesses and reviews pupils' understanding of the learning outcomes.

3. Though a variety of activities is important, each activity must be appropriate to the learning at hand. A variety of activities provides pupils with an opportunity to learn in different ways, and thereby to build up and develop the skills to do so effectively. At the same time, it does not mean that every lesson must involve a variety of activities. It depends on the complexity of the subject matter, the ability and interest of the students.

Example for Learning activities for the initial phase of the lesson, i.e., introduction or motivation.

Example for learning activity for the objective – estimating the value of a given power of a natural number.

T : Hello! My dear students, tell me what is 2^3?

S_1: Oh! $2^3 = 8$

T : How did you get that?

S_2 : I multiplied 2 itself as a factor 3 times.

T : Good. Then what is 2^4?. How do you get the value?. -------- S_3

S_3 : Sir, it is 16., we obtain by multiplying 2 by itself as a factor 4 times.

T : Okay! You tell me the value of $2^{3.6}$? -------- S_4

S_4 : No. How can you find?

How can we multiply 2 by itself 3.6 times?

T : You may not be able to multiply 2 by itself 3.6 times.

Can you estimate the value of $2^{3.6}$?

Can you tell in between which two values does $2^{3.6}$ exist?

S : Since, 2^3=8; 2^4=16

Then probably a number that lies between 8 and 16 must correspond to $2^{3.6}$.

T : So, let us see, how to find $2^{3.6}$?

Example for Learning activity for the objective:

To distinguish between examples and non-examples of circles and explaining the defining attributes of a circle (Cognitive: Conceptualisation).

T : I would like someone to tell us about the racing game we played with the throw ball yesterday. Okay, Mr. Sam.

S : Two of us raced at a time to see who could get the ball first.

T : Draw a picture showing how we first line up and where the ball was.

S : Draws the picture (see Fig. 6.45).

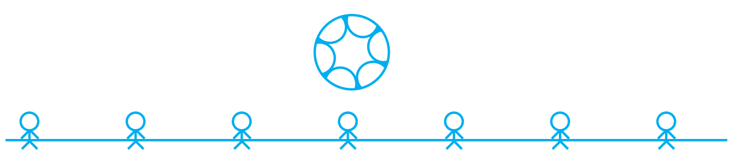


Fig. 6.45

T : Now draw, a picture showing how we lined up after changing the rules.

S : Draws the picture (see Fig. 6.46).

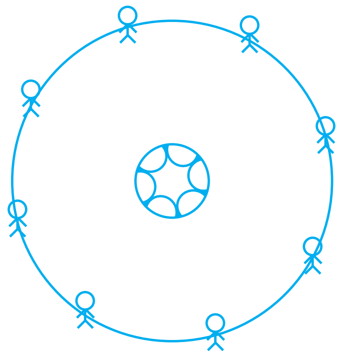


Fig. 6.46

T : Why did we change the rules?

S_4 : Because they were not fair before; some people were closer to the ball than others.

T : Why was the second way fairer?

S_5 : Because everybody was just as close.

T : What shape did you make after we changed the rules?

S_{10} : A circle.

T : Why was that better than being on a straight line?

S_{15} : Because we were around the ball, so that no one was closer or farther away from the ball.

T : So, what is a circle then?

S_5 : A circle is a kind of thing that is all around a ball.

T : Yes, good.

When you are standing on a line some of you are closer and some are farther from the ball.

When you are around the ball and with the same distance from the ball, you make a shape of a circle.

So, a circle is a set of points (like your positions) in a plane which are equidistant from a fixed point.

(e) Assignments

At the end of the lesson plan, after review or summary of the lesson, the assignment follows. The teacher has to carefully plan the assignment to be given to students. It may consist of exercises or problems to be done outside the classroom, i.e., in the hostels or in their houses leisurely. Decisions about the length, type, nature of the assignments should be made while planning the lesson. Assignments that are not clear, confusing or unrealistic will only inhibit learning.

The purpose of assignments is to provide activities in mathematics. Problems and exercises provide a context for applying generalisations and for practising algorithms basic to the development of mathematical skills.

The assignment should be assigned as early as possible after the review is over and when the teacher has the full attention of the students. The teacher should tell the students clearly what is to be done, and how to do it.

With a wide range of abilities, reading, speeds, interests and aptitudes operative in any class, a differentiated assignment can be given. Differentiation does not need to be blatantly obvious. Minimum assignment for all with the suggestions that pupils who find the going rather easy move ahead to problems of a more complex nature.

Assignments act as a barometer for students achievement for the teachers. The teacher can use the student's work in doing assignments as a diagnostic tool. The type of students errors can provide the teacher with valuable information for future lessons, reteaching and constructing follow up activities.

Student's work in doing assignment can be a starting point or can be used as a platform to initiate new learning. Assignments can also be provided with some answers so that students can gain confidence after achieving parts of the assignments, which in turn motivates them to complete the assignments.

(f) Assessment for Learning

Assessment is an integral part of teaching and learning. Planning for a lesson includes preparation of assessment materials that assess the students against the instructional objectives set by the teacher that are intended to be achieved at the end of the lesson.

The assessment can occur in all the three stages of lesson. In the initial phase of the lesson, the teacher evaluates the prerequisite knowledge of the students in order to know whether the pupils are ready to move onto the next stage in the learning process or he/she needs to recapitulate on the prerequisite knowledge to remove any misconceptions or problems being experienced by the pupils.

In the second phase of the lesson questions to the students use of exemplars and non-exemplars, practical activities, such as investigations or problem-solving situations planned in the learning experiences gives feedback to the teacher, a valuable information about the appropriateness of the learning goals to the pupils age and ability; the effectiveness of the instructional methods and the resources that are used.

In the final phase of the lesson, while reviewing the lesson, the questions help students to reinforce the new concepts and skills covered in the lesson.

Many formal and informal ways of evaluation are discussed in Unit 8.

6.3.7 Structure of a Lesson Plan

A comprehensive plan might indicate at the top the essential identification data like the name of the teacher, school, standard, section, subject, topic, period, time, date, etc. The instructional objectives could then be selected and clearly written. What are often stated as general aims could be taken for granted and so need not be repeated lesson after lesson. For example, students will have the knowledge of ...., understands ...., applies ...., etc., could be assumed and need not be specifically stated unless some important aspect of this is to be specifically emphasised in the lesson.

The instructional objectives should be stated in terms of specific teaching objectives, stated in terms of pupils observable behaviour.

They should be followed by the previous knowledge, i.e., background of experiences and understandings, that the students are expected to have, which will facilitate to link the new knowledge, the concepts, the generalisations which the teacher is supposed to impart on that day.

Then under the head of teaching points, a mention is needed to be made about the concept names, concept definitions, the statements of the generalisation, etc., that the teacher is going to teach.

The learning aids, materials, etc., to be used could also be indicated immediately after teaching points.

Since all teaching and testing have to be objective based and learner-centred, due emphasis should be given on the student behaviour in the process of learning as well as in the product to be formally tested, the plan should clearly show such learning outcomes in terms of pupils observable behaviours.

These then should be the starting points for indicating the corresponding content, learning activities, evaluation devices and items, etc. Such things could be given in a structured way. *There is no rigidity about the number or different order of columns,* but a good comprehensive plan adopting this approach should essentially indicate the important expected learning outcomes, sequential learning activities and an evaluation procedure. The content need to be spelt out in detail and every question and also every small details be given under learning activities so that the teacher (atleast a new teacher) finds himself at ease in actual classroom. As far as possible the learning activities could be given from the pupils point of view indicating the teachers role by implication. Questions for testing the product outcomes could be given to see whether the stated objectives have been attained by the pupils.

In addition, the plan should include a review or a summary followed by an assignment.

All these could then be given in a structured way in about three columns with one-to-one horizontal relationship.

6.3.8 (a) Example of Lesson plan

Topic : Algebra	Date : __________
	Class : XII
	Period : __________
Unit : Equivalence Relations	Time : 45 minutes

Instructional Objectives

At the end of the lesson, a student will be able to:

1. define an equivalence relation
2. state the characteristics of an equivalence relation
3. identify equivalence relation from the given set of relations
4. cite examples of equivalence relation
5. relate equivalence relation with other type of relations.

Teaching Points

Equivalence relation on a set is a relation on the set which satisfies Reflexive Property, Symmetric Property and Transitive Property.

Previous Knowledge

Reflexive, Symmetric and Transitive Relations

Expected Learning Outcomes	Sequential Learning Activities with Inbuilt Evaluation	Blackboard Work
	Introduction	
	T: Good Morning students (seeks the attention of the class). We have seen some properties of a relation and based on these properties, we have distinguished between types of relations	
	T: What are the properties of a relation? … S_1	
Recalls and lists	S_1: Reflexive, symmetric, transitive, anti-symmetric.	
	T: Good. Mention the types of relations. …S_2	
	S_2: Reflexive, symmetric, transitive, anti-symmetric.	

<table>
<tr><th>Expected Learning Outcomes</th><th>Sequential Learning Activities with Inbuilt Evaluation</th><th>Blackboard Work</th></tr>
<tr><td></td><td>T: I have given you some homework to write the examples of reflexive, symmetric and transitive relations. Give me an example of reflexive relation. ….S_3</td><td></td></tr>
<tr><td>Gives an example of a reflexive relation.</td><td>S_3: “is equal to” in a set of R.
T: Does this binary relation satisfy any other properties? …S_4

S_4: Symmetric and transitive.

T: Good, in the same manner check the other relations. (Give some time to the students). What about the other binary relation “is perpendicular to”?…..S_5

S_5: It satisfies only symmetric property.

T: Repeat the same for few more binary relations (“is less than”, “is parallel to”).

Development of the Concept

T: Look at the binary relations (showing to chalkboard). Are there any commonalities between the binary relations “is equal to”, “is parallel to” etc. and other binary relations? ….S_6</td><td><table>
<tr><th>Binary Relation</th><th>R</th><th>S</th><th>T</th></tr>
<tr><td>“is equal to”</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>“is perpendicular to”</td><td>×</td><td>✓</td><td>×</td></tr>
<tr><td>“is less than”</td><td>×</td><td>×</td><td>✓</td></tr>
<tr><td>is parallel to</td><td>✓</td><td>✓</td><td>✓</td></tr>
</table>
R- Reflexive
S- Symmetric
T- Transitive</td></tr>
<tr><td>Compares and contrasts the binary relations which satisfy all the three properties with other binary relations.</td><td>S_6: “is equal to”, “is parallel to”, etc. are satisfying all the three properties when compared to other binary relations.</td><td>Teacher encircles (or highlights) those binary relations which satisfy all the three properties of equivalence relation.</td></tr>
<tr><td></td><td>T: Good. The binary relations which satisfy all the three properties are called equivalence relations (writes on the board).</td><td>Equivalence Relation is a binary relation which satisfies reflexive property, symmetric property and transitive property.</td></tr>
</table>

Expected Learning Outcomes	Sequential Learning Activities with Inbuilt Evaluation	Blackboard Work
	T : What is required for a binary relation to be an equivalence relation? ….S_7	
States the sufficient condition.	S_7 : It should satisfy reflexive, symmetric and transitive properties.	
	T : Good. (Gives a binary relation "is less than or equal to" and asks). Is this an equivalence relation? ….S_8	1. "is less than or equal to" in a set of Real numbers. 2. "is to the left of" in a set of points on a line. 3. "is collinear with" in a set of points in a line.
	S_8 : Is "≤" not an equivalence relation?	
	T : Why do you think so? ….S_9	
Gives reason for the binary relation as the non-example of equivalence relation.	S_9 : It does not satisfy symmetric property.	
	T : Good. Give me an example of an equivalence relation? …. S_{10}	
Gives an example of equivalence relation.	S_{10} : "Has same perimeter as" in the set of all triangles.	
	T : Why is it an equivalence relation? ….S_{11}	
Gives reasons for a binary relation to be an equivalence relation.	S_{11}: Since it satisfies all the three properties.	
	T: Right. We have seen that there are three conditions for a binary relation to be an equivalence relation.	1. Reflexive, 2. Symmetric and 3. Transitive properties.

Review and Evaluation

What is an equivalence relation?

What are the characteristics of an equivalence relation?

An equivalence relation is a kind of .... relation.

What similarities and differences do you find between equivalence relation and other relations?

Today we learnt about equivalence relation.

Strategy

Compare and contrast → Definition → Sufficient condition → Non-example → Necessary condition → Example → Sufficient condition.

6.3.8 (b) Example of Lesson Plan

Topic : Linear equations in two variables	Date :
	Class : IX
	Period :
Unit : Algebra	Time : 45 minutes

Instructional Objectives

At the end of the class or lesson, students will be able to:

(i) state the definition of linear equations in two variables

(ii) state the characteristics for an equation to be linear equation in two variables

(iii) cite examples of the linear equations in two variables

(iv) identify linear equations in two variables

(v) formulate linear equations in two variables.

Previous Knowledge

(i) Definition of linear equation in one variable

(ii) Formulation of linear equation in one variable.

Teaching Points

Linear equation in two variables is an equation that contains two different variables each of degree one.

Expected Learning Outcomes	Sequential Learning Activities with Inbuilt Evaluation	Evaluation/Blackboard Work
	T : Good morning S_3 : Very good morning, Sir! T : In your earlier classes you have learnt linear equations in one variable. Give a few examples for linear equations in one variable.	
Gives examples for linear equations in one variable.	S_{10} : $x + 5 = 0$ $y - 2 = 0$ $2z = 6$	$x + 5 = 0$ $y - 2 = 3$ $2z = 6$
	T : Why are these called linear equations ?	
States the necesary condition for equations to be linear.	S_{15} : These are linear equations because, the degree of the variable is 1.	
	T : How many variables are there in each of the equations	
Identifies the number of variables in the equations.	S_6 : One	
	T : Ram and Rahim together have 25 books, whereas Rahim has 5 more books than Ram. Find the number of books Ram and Rahim have separately? S_9 : Let x be the number of books Ram have. So Rahim will have $x + 5$.	
Solves the problem by formulating the linear equation in one variable.	Therefore, $x + x + 5 = 25$	

or, $2x + 5 = 25$

or, $2x = 25 - 5 = 20$, i.e., $x = 10$.

So, Ram has 10 books and Rahim has (10 + 5) books = 15 books.

T : Good. See here on the blackboard. A set of equations given in one column as set A and in other column as set B (Table 1).

Table 1

Set A	Set B
1. $x + 2y = 3$	$x - y + z = 8$
2. $2r + 5s = 0$	$n = 5$
3. $u = 2v + 4$	$u + v + t = 0$
4. $x - 2z = 6$	$2x + y + u + 3 = 5$
5. $3t = 5u - 8$	$x + 8u = 3z$

All the equations in set A have some thing in common which is not found in the equations of set B. Observe and tell me what is that?

Compares each of the equations with others in set A and contrast with equations of set B and states the findings.

S_7 : Equations in set A contain two variables whereas those in set B, have the number of variables either 1 or more than 2.

T : Yes, good. In set A, each of the equations contains two variables.

T : Now observe the equations in Table 2 and see that the equations in set A has something in common which is not found in every equation of the set B. Try to find out that.

Table 2

Set A	Set B
1. $x + 2 = 0$	$x^2 - 5 = 3$
2. $x + y = 5$	$x^2 + y^2 = 4$
3. $u = 2v + 4$	$r + 2s^2 = 0$
4. $s - 2t + 5$ $u = 6$	$r^2 + 2t - u^3 = 9$

Compares and contrasts and finds the commonalities in set A.

S_{10} : All the variables in each of the equations given in set A are of degree 1, while it is not so in the equations in set B.

	T : Good. Now any one of you come to blackboard and write equations of degree 1 which contain two variables.	
Writes the equations	S_2 : (Writes) $x + y = 5$ $2t + 5 = 3u$	
	T : Good. What do you call an equation containing variables of degree 1?	
Recalls from the previous knowledge	S_3 : It is a linear equation.	
	T : What is a linear equation in two variables?	
Tries to define	S_5 : A linear equation in two variables is an equation that contains two variables.	
	T : Then, is $x + 2y^2 = 5$ a linear equation in two variables?	
Recognises the lack of necessary condition.	S_1 : No, because one of the variables has degree 2.	
	T : Can any one restate the definition given by S_5?	
Modifies the definition given by S_5.	S_6 : A linear equation in two variables is an equation that contains two variables each of degree 1.	
	T : Yes good. So, a linear equation in two variables is an equation having two variables, each of degree 1.	
	T : give two examples of linear equation in two variables ....S_7	
Gives examples.	S_7 : (1) $15u - t = 0$	

	(2) $x - 8y = 5$	
	T : Good. (Write an equation on the black board) Is this a linear equation in two variables ? S_5 : No T : Why? S_5 : Because it does not contain two variables.	$x = 8$
Identifies the lack of necessary condition.	T : Is this a linear equation in two variables ? S_8 : Yes. T : Why?	$t + 5u = 8$
Identifies the sufficient condition for an equation to be linear in two variables.	S_5 : Because it contains two variables and also each of the variable is of degree 1.	
	T : Is this a linear equation in two variables ? S_5 : No T : Why,	$t^2 = 5u$
Identifies lack of necessary condition.	S_5 : It is not linear. T : Yes good. For an equation to be linear it should contain two variables and also each variable should have degree 1.	
List the linear equations in two variables and others.	T : (Writes a set of equations on the blackboard) Write down the linear equations in two variables and the ones that are not. S_6 : Writes.	(1) $x + y = 8$ (2) $2x^2 - y = 4$ (3) $2t + u = 10$ (4) $x^2 - u^2 = t^2$ (5) $x^3 - y^3 = 8$ (6) $x + y = 2t$

	Linear equations in two variables are 1, 3, 7, 10 and others 2, 4, 5, 6, 8, 9 are not the linear equations in two variables. T : Miss Kalyani has a few hen and a few pigs in her farm. Total legs of the hens and pigs is 50 and the heads is 17. Can you express these in the form of equations? S : (Silence) T : Is one variable enough to represent the situation? S : No. T : Then, how many variables are necessary?	(7) $s + 2u = -8$ (8) $r + s^2 = t^2$ (9) $a + b = c$ (10) $5a + 6b = 9$
Identifies the variables	S : May be, we require two variables – one for pigs and one for hens. So, let the number of hens be x and the number of pigs be y. T : Then how many heads each one of the hens and the pigs has? S : Each one has one head. T : How many heads x hens and y pigs have?	
Represents in the form of an equation.	S : $x \times 1 + y \times 1 = x + y$ But it is given equal to 17 Therefore, $x + y = 17$	

	T : Now can you represent the legs of the hens and pigs in an equation form?	
	S : $2x + 4y = 50$.	
	T : Each student in Class IX has 4 textbooks and 6 note-books and each student in Class V has 2 textbooks and 2 notebooks. Total number of textbooks is 100 and the notebooks is 140. Can you represent this information in the form of equations ?	
Represents the data in the form of equations.	S : Let x be the number of Class IX students and y be the number of Class V students, then $4x + 2y = 100$ $6x + 2y = 140$	
	Review: In a right angled triangle, write the equation representing the sum of the other two angles other than the right angle. What are the variables ?	
	Assignment : Prabhakar has some notes of Rs 5 and a few notes of Rs 10. Total value of the money is Rs 125. Can you express this as a linear equation in two variables ?	

EXERCISE 6.2

Prepare a lesson plan on each of the following:

(a) Rhombus

(b) Skew lines

(c) Percent

(d) Profit and loss

(e) The roots of a quadratic equation

(f) If two lines in the same plane are intersected by a transversal forming equal alternate angles, then the lines are parallel

(g) Pythagoras Theorem

6.3.9 Use of ICT in Teaching-Learning Process

ICT (Information and communication technologies) has the potential to make a significant contribution to pupil's learning in mathematics by helping them to:

(i) practice and consolidate learning skills by using software to revise and to give rapid assessment feedback.

(ii) develop skills of mathematical modelling through exploration, interpretation and explanation of data by choosing appropriate graphical representations for displaying information from a data-set; by experimenting with forms of equations in trying to produce graphs which are good fits for data-plots; by using a motion sensor to produce distance time graphs corresponding to pupils own movements.

(iii) experiment with, make hypothesis from, and discuss or explain relationships and behaviour in shape and space and their links with algebra, by using software to

(a) automate geometric constructions

(b) carry out specified geometric transformations

(c) perform operations on co-ordinates or draw loci.

(iv) develop logical thinking and modify strategies and assumptions through immediate feedback by planning a procedure as a sequence of instructions in a programming language or a sequence of geometrical constructions in geometry software or a set of manipulations in a spreadsheet.

(v) make connections within and across the areas of mathematics; for example: To relate a symbolic function, a set of values computed from it, and a graph generated by it to a mathematical or physical situation, such as the pressure and volume of a gas, which it models.

(vi) work with realistic and large sets of data.

For example: carrying out experiments using large random samples generated through simulation.

(vii) explore, describe and explain patterns and relationships in sequences and tables of numbers; by entering a formula in algebraic notation to generate values in an attempt to match a given set of numbers.

(viii) learn and memorise by manipulating graphic images. For example, the way the graph of a function such as $y = x^2$ is transformed by the addition or multiplication by a constant.

ICT also has the potential to offer valuable support to the mathematics teachers by:

(a) helping them to prepare teaching materials

For example: Downloading materials for classroom use from the internet, such as mathematics problems for pupils to solve with accompanying teachers notes, software for computers and reviews of published resources.

(b) providing a flexible and time saving resource that can be used in different ways and at different times without repetition the teachers input by enlarging fonts, adding diagrams or illustrations, adapting parameters used in problems.

(c) providing a means by which subject and pedagogic knowledge can be improved and kept up-to-date by accessing the virtual teacher centre to obtain practical advice, to exchange ideas with peers and 'experts' outside school.

(d) aiding record keeping and reporting by storing and regularly updating formative records which can form the basis of a subsequent report.

Summary

The content of mathematics is divided into several Units based on themes or ideas. A Unit includes all the materials that covers a theme. Each Unit is further subdivided into several lessons. For the learning to be effective, teacher has to visualise the entire teaching - learning situation that is likely to develop in the classroom and plan accordingly. The lesson plan includes the specific objectives, i.e., the intent of the teacher that a student will be able to do at the end of the instruction; previous knowledge, that the student is expected to possess that facilitates the new learning; learning experiences, the activities undertaken by students planned by the teachers and use of instructional aids with a specific purpose to bring about a behavioural change in the students.

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The tantalising and compelling persuit of mathematical problems offers mental absorption, peace of mind, endless challenges, repose in activity, and battle without conflict, "refuse from the goading urgency of contingent happenings", and the sort of beauty changeless mountains present to sense tried be the present-day kaleidoscope of events.

Morris Kline

UNIT
7

LEARNING RESOURCES IN MATHEMATICS

7.1 Introduction

A resource may be known as a source of aid or support that may be drawn upon when needed. Learning of a subject widely depends upon the availability of resources which plays a vital role in strengthening the understanding of the subject matter and hence, the fundamentals of the subject itself. While in life sciences and humanities, there are physical evidences available, viz., plants and animals in biology; people, laws, countries, social and cultural context in humanities, but in mathematics, evidences are normally invisible. Unlike other subjects, mathematics is basically based on logical reasoning and generalisation and deals with abstract concepts; thereafter their representations through symbols etc. Further conceptual building on these abstract concepts bring greater vigour and complexity to its structure. So, there is a crucial need for learning resources in mathematics as they are not directly available.

Resources, here, are anything which may assist in understanding, strengthening or extending the subject content. Resource can also be understood as something that one uses to achieve an objective. In case of mathematics learning, these may be textbooks, handbooks, supplementary books, media, visual, audio, audio-visual, community, locality, school, infrastructure, mathematics laboratory, mathematics exhibitions, excursions, tours and trips, mentors, computer softwares, internet, web 2.0 tools etc. There is a basic need to explore more of these resources which may help in learning of mathematics, make it interesting, removing mathematics phobia from minds of learners, and hence, achieving narrow aims as well as higher aims for school education as mentioned in Teaching of Mathematics Position Paper, NCF–2005. Pimm and Johnston (2005) mention that using learning resources in mathematics, pupils are engaged through stimulating curiosity, communicating enthusiasm, matching the approaches used to mathematics being taught, effective questioning, selecting and making good use of resources and exploiting opportunities to contribute to the quality of pupils' wider educational development. This is the need of the hour as far as mathematics learning is concerned, that equal importance should be given to urban as well as to rural area students. This will strengthen our way, as teachers, in achieving the twin premises of mathematics education, that all students can learn mathematics and that all students need to learn mathematics (NCF– 2005). While talking about school mathematics, Romberg (1999) says, for learning of mathematics two problems are required to be dealt with. First, the content and structure of the curriculum should not operate to indoctrinate students with past values, but should be derived from visions of the future. All students should be taught to reason, to design models and to create and solve problems. Second, all children must be taught critical thinking skills. He/she emphasises to press the need for all citizens to become 'mathematically literate'. This can be achieved with the help of better learning resources in mathematics. Conducive to the spirit of mathematics, NPE (1986) reflects, "Mathematics is a vehicle to train a child to think, to reason and articulate logically." In this Unit, we shall discuss various learning resources in mathematics.

Learning Objectives

After studying this Unit, student-teachers will be able to:

- develop the understanding of concept, need and significance of the textbook as a meaningful resource in learning of mathematics.
- develop the understanding of concept and need of multimedia and audio-visual multimedia as a meaningful resource in mathematics learning.
- develop the understanding about criterion for selection of an appropriate multimedia resource for a chosen content area.
- develop the ability to design learner-centred multimedia resource to meet specific learning needs of the learners.
- develop the understanding of concept of community resources and pooling of learning resources in mathematics at school level, block level and district level to strengthen mathematical concepts.
- develop the understanding of social, ethical and technical hurdles and issues related to handling these hurdles in utilising learning resources in mathematics.

7.2 Textbook

Fig. 7.1

The most common learning resource in mathematics is a 'textbook.'

Various people conceptualise textbook in various ways. Some of these are:

A textbook is a book used in schools or colleges for the formal study of a subject.

Another statement describes it as *a book used as a standard source of information on a particular subject.*

A textbook is a collection of series of texts on various concerns of a specific area, e.g., a textbook on mathematics will be consisting various topics in mathematics specific to a particular grade and comprising specific text on these topics. A textbook of Class X may differ in various aspects from a textbook at college level for the same subject. In Class X textbook, 'trigonometry' may be one of the several chapters while there may be a complete textbook on 'trigonometry' alone at college level.

Textbook may vary according to the subject, content, student's age level etc. It gives introduction of the content and then tries to inculcate the understanding of the content through various instances, fundamentals, examples, exercises, etc. A textbook should preferably have historical context too, as it sensitises learner to humanistic aspect of mathematics. As Lawrence (2006) reveals that the historical context offers a flexible framework within which it is possible to achieve good results. It may have story also to develop communication skills, empathy, understanding and above all the subject knowledge.

7.2.1 Need and Importance

From the above discussion, it can be said that a textbook is a very important learning resource as it not only introduces the content, but also builds a platform on which the entire structure of the concept and content knowledge stands firmly. It also gives a framework of a particular area of a course of study. It may be made easily available to all students irrespective of their socio-economic background. Now a days, textbooks are available in hard as well as soft forms. It can be in the form of a paper book or an e-book.

7.2.2 Handbook and Reference Book

Fig. 7.2

Apart from the textbook, there are other type of books, such as handbooks and reference books. A handbook is a complete book in concise form on a particular task, profession, or area of study etc. A reference book may consist of details or further explanation on a particular topic of a textbook, extension of the topic, further examples/problems and further suggestive texts. Such books may be used to supplement to any resource and also to expand and strengthen the content. As an example, the book 'The Mathematics of Egypt, Mesopotamia, China, India and Islam : A Sourcebook' edited by Victor J Katz (2007) can be used as a reference book to know historical development in mathematics. In continuation of handbooks and reference books, supplementary books, viz., 'Exemplar Problems in Mathematics' and 'Laboratory Manuals in Mathematics' by NCERT for various classes, may also be seen as good learning resources.

7.3 Audio-Visual Multimedia

Basically, there are five senses: to see, to hear, to smell, to taste and to touch. These are perceived with the help of different body parts like eye to see, ear to hear, nose to smell, tongue to taste and skin to touch. But for all these, a medium is must, e.g.– to hear 'a sound should' be there, to see 'some image' or picture should be there.

In case of teaching-learning process, we can talk of various media like picture, image, sound, still and moving clip art, silent and audible video. *'Multimedia' is the combined use of media, such as movie, music, lighting, CD-ROMs and the internet for education or entertainment.*

So, *a multimedia can be understood as the combination of more than one type of media,* e.g.– sound alongwith image. *When we use audio as well as visual media simultaneously, it is called audio-visual multimedia*. It can also be viewed as *multimedia designed to aid in learning or teaching by making use of both hearing and sight, for example*, a video clip with audio, a movie clip, a power point presentation with sound effects, a video film or a television programme. Multimedia is a form of Information and Communication Technologies (ICT) (Cohen, Manion, Morrison, 2005). There are many ways in which ICT can be used to enhance the teaching-learning experience in mathematics. Some of the educationists are more concerned with the efficiency and attitude of mathematics teachers towards use of multimedia and ICT. Crisan, Lerman and Winbourne (2007) conducted a study and concluded two types of factors – firstly factors pertaining to contextual nature and secondly factors pertaining to personal nature, if dealt and facilitated effectively, can help to the integration of ICT into teaching of mathematics at secondary school level. Factors of contextual nature are school context, availability and accessibility to ICT tools and resources, teacher's ICT skills, department ethos and key persons and teacher's ICT professional development. Factors of personal nature are ICT content conceptions, ICT curricular conceptions, conceptions in mathematics, pedagogical content, conceptions and teacher's own learning experiences with ICT.

7.3.1 Why Multimedia?

When we use more than one media simultaneously, it helps in learning better than just one media. When audio-visual multimedia is used, two senses work together: to see as well as to hear. So, learning in this case is acquired through coordination of both the senses. In this way, more than one sense can be activated to make effort to understand the content simultaneously. It may save time too than using single media at a time. Interaction of different senses help in linking of various information simultaneously and thereby stimulates the process of concept formation which is essential for the formation of an abstract concept, e.g.– if a student observes a lion's picture (visual) alongwith its voice (audio) and if repeated a number of times, the student may recall either when presented one of the two. Similarly, when a student observes a rupee five note (visual) alongwith somebody pronouncing as five rupees (audio), she/he would learn better.

Traditionally, we are using materials like chalk, blackboard, simple paper charts, mathematical geometry box, etc. in teaching-learning of mathematics. Multimedia has opened a new window, as far as, teaching-learning aids are concerned. Use of multimedia in teaching stimulates our senses for better coordination and metacognitive processes. There are various advantages of using multimedia than that of traditional aids. CALtoonz2006 looks at it as follows:

Multimedia can

(i) animate the static

(ii) simulate the hazardous or costly experiment

(iii) capture reality

(iv) add movement to static concepts

(v) add dimension to abstract concepts

(vi) add an element of fun in sometimes boring situations

(vii) include audio/video clips of the original person/event, such as the speech of eminent mathematicians and educationists.

Multimedia can add effects, create virtual situations, arouse interest among learners, give concrete representation for learning of abstract concepts, use pictures, texts, images, audio, video, animations, etc. It adds beauty and variety to mathematical concepts. So, it can be easily concluded that use of multimedia can be highly advantageous for the learning of mathematics as it tries to transform its abstractness into sort of presentable concreteness. It helps in making mathematical concepts more real and accessible to learners.

7.3.2 Multimedia in Mathematics

There are various organisations and bodies which are producing audio-visual multimedia packages or video CDs in the field of mathematics, viz. Central Institute of Educational Technology (NCERT), Electronic Media Production Centre (IGNOU), and different Government Directorates/Departments of Education. For instance, CIET has produced multimedia packages like 'Mathematics for Secondary Classes : Locus' and 'Mathematics for Primary and Upper Primary Classes : Construction of Geometircal Shapes', etc. It has produced many audio and video programmes both for teachers and learners. There are some non-government organisations (NGOs) and private sector educational institutes too which produce and make available the audio-visual multimedia for mathematics. It can be effectively used even in the absence of teacher, any number of times with a freedom to manage own time schedule. This ultimately enables learners to become independent learners.

7.3.3 How to Select a Multimedia?

Whatever is the content or whatever be the media, learner is always our main concern. So, while selecting any multimedia, the learner should be at the central place.

There are several aspects which should be kept in mind while selecting an appropriate multimedia:

1. *Pertaining to the learner*
   - (i) Learner centredness: It should be the learner who actively participate and takes decisions during execution of content through multimedia.
   - (ii) Motivation and encouragement: Learner should get motivated and encouraged by the multimedia to be used.
   - (iii) Readiness and mental set: Learner should be made ready and mentally prepared to learn from multimedia to be used.
2. *Pertaining to the multimedia material*
   - (i) Usefulness: The content of multimedia should be useful for the learner.
   - (ii) Cost effectiveness: Multimedia should be cost effective.
   - (iii) Availability: Multimedia should be easily available to all.
   - (iv) Relevance: Multimedia should be relevant to the subject and topic.
   - (v) Accessibility: Multimedia should be easily accessible to all.
   - (vi) Duration: The duration of multimedia should be appropriate with respect to need of content topic and learner. It should be neither too short nor so long.
3. *Pertaining to the formal and non-formal setting*
   - (i) Physical conditions : Physical environmental conditions should support the multimedia to be used.
   - (ii) Apparatus and equipments : Availability of apparatus and equipments for multimedia should be kept in mind.
   - (iii) Learner friendliness : The setting in which multimedia is to be used should be learner friendly.

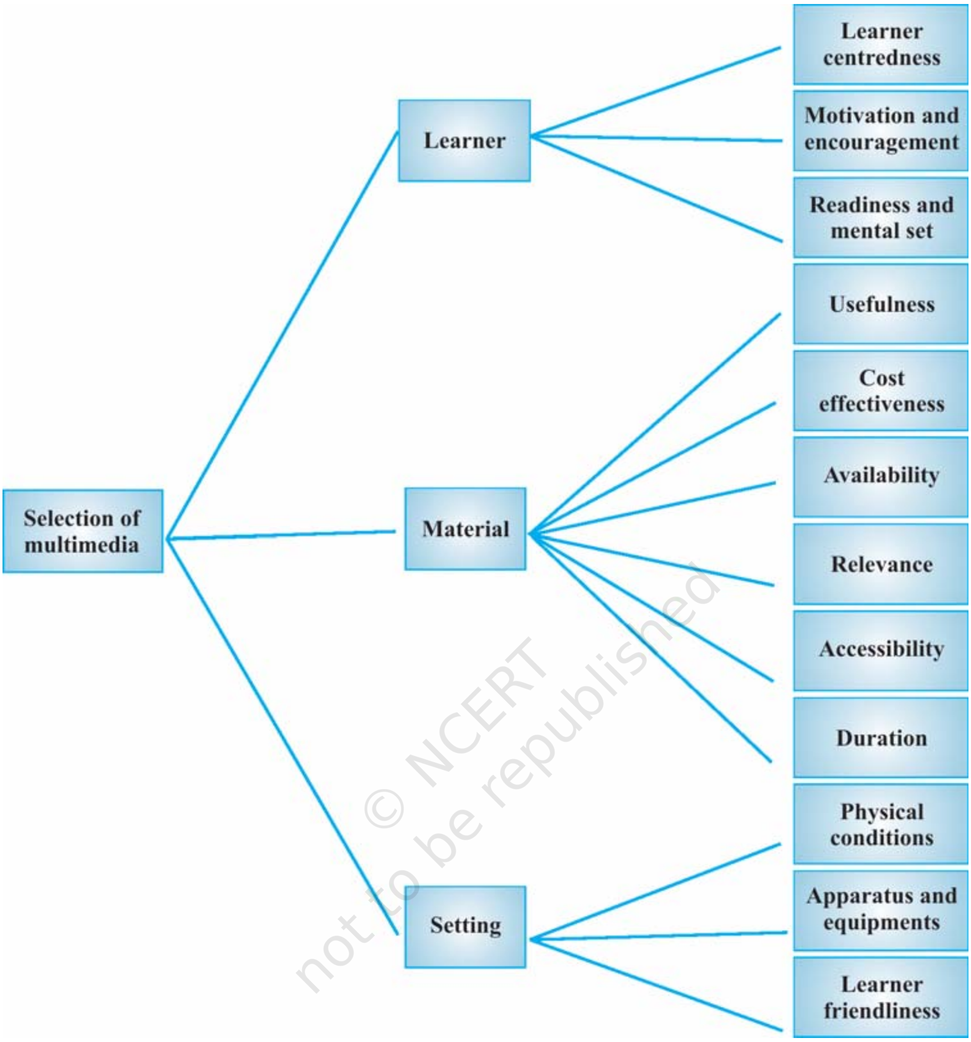

Fig. 7.3

7.3.4 How to Design a Multimedia?

Although there are variety of multimedia resources available in the market, it is always better if a teacher designs a multimedia resource keeping in mind the learning demands of the learners. Such individualised resources will have better impact and results. A multimedia can be designed through systematic steps, alongwith steps mentioned by various stakeholders, as follows:

(i) Defining end point behaviour

(ii) Assessing the learner or learner group

(iii) Defining specific objectives

(iv) Planning materials and objectives

(v) Planning optimum utilisation of available resources and alternative arrangements

(vi) Planning teaching strategies and methods

(vii) Evaluation process and recycling

Directorate of Education, Government of National Capital Territory (NCT) of Delhi along with Earnst Young Foundation (EYF) launched a project CALtoonz2006 for preparing multimedia for school students. It gave a sequence of steps for multimedia content development as follows:

(i) Defining the learning objectives

(ii) Gathering information

(iii) Preparing information for steps of preparation

(iv) Preparation of material for guided and independent practice which includes:

(a) Database of questions

(b) Games

(v) Art work and animation

(vi) Voice over

(vii) Finishing touches

(viii) Review and field test

These above mentioned steps can be used to design a good multimedia.

7.4 Community Resources

7.4.1 What is Community?

Community, the term has evolved from the old French word etymologically 'communite' and classical Latin word 'communitas' or 'communis' which means common.

As revealed from various sources, *Community is a group of people sharing a common understanding who reveal themselves by using the same language, manners, tradition and law. It can be seen as the condition of having certain attitudes and interest in common.*

It can also be viewed as *a particular locality, considered together with its inhabitants or a group of people within a society with a shared ethnic or cultural background,* *especially within a larger society.* It also gives a sense of *the people with common interests living in a particular area.* So, as a whole it can be understood as a society in general.

National Curriculum Framework for Teacher Education (2009) emphasises upon the role of community knowledge in education. It says, "It is important for the development of concepts in children as well as the application of school knowledge in real life, that formal school knowledge is linked with community knowledge." Community knowledge, here, refers to the knowledge that people construct, develop and amass as a result of their every-day and ecological experiences. So, indigenous and local knowledge should be given ample importance for the learning.

7.4.2 Mathematics Learning begins at Home

The best initial resources for mathematics learning can be found at home and immediate environment itself. The home, as first exposure of informal community to the child, provides ample of opportunities where mathematical ideas are experienced, explored and learnt. There are various objects available at home and that too with ease, which can help in learning of mathematics. The currency notes can be used to develop the understanding of number system. These can be used to learn fundamental operations in mathematics. The walls, floor, ceiling, electrical wires, table, chair, utensils, etc. can be used to develop spatial understanding and formation of geometrical shapes. Wall and gate can be used to learn trigonometry. Playing cards and glass balls can help in learning of probability. Distribution of biscuits and chocolates, etc. can make the learner to understand the basic operations in statistics like presenting information in terms of numbers and making a frequency table. Not only this, even rituals at home can be used as learning resources in mathematics. There are many books written by Qvarsell and Wulf (2003) mentioning how rituals can be used for learning.

7.4.3 What are Community Resources for Mathematics Learning?

Community in a generalised sense can be viewed as the society. The local context of the learner can be viewed as the best resource for learning. Community resources give an opportunity for better inquiry system for learning.

For developing and schematising concepts through various sources of evidence like reasoning, observation, representation, dialectic and ethical values, community resources can play a vital and facilitating role. Students are advised to explore more upon these sources of evidence.

At formal stage, school, block and the district, as parts of community, play significant role in shaping mathematical world of the learner. At all these levels, various resources are there which can help in learning of mathematics. At school level, these may be school premises and school buildings, corridors, verandahs, classrooms and walls, mathematics laboratory or corner, mathematics club or forum, group of mathematics teachers or mentors and mathematics exhibitions. Group coordination and cooperation among mathematics teachers can help better teaching-learning of mathematics and improve performance in mathematics as revealed by Horn (2008). At block/cluster level, there may be cluster level exhibitions, cluster level mathematics centres, a panel of all mathematics teachers at block level, block level mathematics competitions. At district level, there may be mathematics centres, mathematics exhibitions, mathematics fairs, committee for mathematics activities, mathematics laboratory at District Institute of Education and Training (DIETs). This is how, we can create opportunities for sharing or exchanging experiences and ideas among thinking community of mathematics practitioners and learners.

7.5 Pooling of Learning Resources

As our everyday experience, we find various concrete and abstract sources of learning, all around us, which act as learning opportunities for meaningful construction of knowledge. One can find everywhere around her/him the sources for learning, and everything (either concrete or abstract) around the learner can make the learner experience so many concepts. Mathematics learning too is not an exception in this case. Though mathematics learning opportunities and resources are available in plenty as part of our everyday life, still there is a need for pooling these resources in some organised and orderly formal and informal manner, so that they become more accessible and transparent. This pooling can be ensured at various levels viz., school level, block level, district level, state level, national level and international level. Pimm and Johnston (2005), while talking about resources and ideas for enhancing the teaching-learning of mathematics, enlist various resources like textbooks and schemes, practical apparatus, homework, parents, learner's room, the history of mathematics, role play, simulation, video and television, school libraries, mathematics clubs and trails.

At national and international levels, science centres, museum, seminars, conferences, symposium, journals, teachers' association can be good resources for mathematics learning. These will be later dealt in Unit 10 of this book. Here, we will talk about learning resources at:

(i) School level

(ii) Block level

(iii) District level

Let us discuss these in detail.

7.5.1. School Level

7.5.1.1. Mathematics Laboratory/Corner

Fig. 7.4

In every school, a mathematics laboratory or corner can be established, which will have various equipments, apparatus, charts, models : working and static, etc., that can help in building the learning of abstract concepts in mathematics by having experimentation, activities, hands on experience, verification, etc. In mathematics laboratory electronic calculator, graph machines, mathematical games, puzzle boards, mathematical kit, mathematics videos and clinometers, etc. can be made available. Mangal and Mangal (2009) has given an elaborated list of hardware instructional aids, viz., magic lantern, epidiascope, projector, radio, tape recorder, television, closed circuit television, video cassette recorder, motion pictures, computers and software instructional aids, viz., blackboard or chalkboard, bulletin board or information board, flannel board, pictures, charts, graphs, maps, globes, diagrams, photographs, cartoons, posters, newspapers, flash cards, models, slides, filmstrips, transparencies, programmed learning packages, many of which can be a part of mathematics laboratory. The need is to think how these can be used for better learning of mathematics.

As NCF – 2005, too mentions that one of the important aims of mathematics education is "to develop the child's resources to think and reason mathematically to pursue assumptions to their logical conclusion and handle abstractions." Mathematics laboratory or corner can best develop the habit of thinking, reasoning and rationalising through logical conclusions and handling abstractions.

Fig. 7.5

7.5.1.2. Mathematics Club/Forum/Community/Society

Fig. 7.6

In school, a club of mathematics students can be established under the guidance of mathematics teacher. The attention should be more towards ensuring membership to all, especially to them who are not thought to be good in mathematics.

Various sorts of activities, discussion, quiz at school level, mathematics excursion and tour, lectures by experts, workshops, competitions can be organised and coordinated by such a club/forum. It can be in the form of a club or forum, community or society. Various activities are mentioned by Thomson and Hartog (1993) in 'Activities to teach mathematics in the context of environmental studies' pertaining to number and number relationship, computation and functions, algebra, statistics, probability, geometry and measurement.

The students who are good at linguistics, too can be motivated to establish a reading club and they can meet weekly to discuss the beauty, nature and recent development in mathematical concepts and hence, it will help in mathematics learning. Group projects can be taken in such a forum which can pave the way for better learning of mathematics with a shift from independence to interdependence, from structured to freedom, from disciplinary to interdisciplinary and from product to process.

7.5.1.3. School Library

Fig. 7.7

Fig. 7.8

School library can be visualised as one of the prominent learning resource for mathematics. There are various textbooks, reference books, activity books and puzzle books that can be made available in the library. These books can be issued to students. Various journals pertaining to mathematics learning can be put inside the library for awareness about mathematics learning and pedagogy of mathematics teaching.

7.5.1.4. Mentoring

Though mentoring is still a developing concept in Indian context, but it can be effectively used as one of the learning resources in the school.

As revealed from the various sources, *a mentor is an experienced person, trusted counsellor or guide who provides information, advice, support and encouragement to a less experienced person, often leading and guiding by example of his/her success in an area.*

Working definition of mentoring can be, "*A one to one learning relationship between an older person and a younger person for the development of the later.*"

Mentors help in strengthening academic skills in general leading to student's success.

Mentoring is a structured one to one relationship or partnership that focuses on the needs of the mentored participant.

Daloz (1990) views effective mentorship as similar to "guiding the student on a journey at the end of which the student is a different and more accomplished person. In a formal learning situation, mentoring functions can be understood as providing support, challenge and vision."

During mentoring too, formal and group work can be given ample emphasis. Mac Bean, Graham and Sangwin (2004) had a study on school and university students namely 'Group work in mathematics: A survey of students' experiences and attitudes' and concluded that students show a very positive attitude towards group work, but with utilitarian view of its benefits.

7.5.2. Block Level

7.5.2.1. Interschool Collaboration

There can be collaboration among schools to provide a place to establish a platform for mathematics. This collaboration will give rise to opening of new opportunities for mathematics learning. This collaboration may be at two different levels: Student's level and Mathematics Teacher's level. Studies have revealed that collaboration of mathematics teachers gives rise to better understanding of learners and learner's problems in mathematics. They discuss various problems pertaining to pedagogy, methods, fundamental problems and sharing available facilities.

7.5.2.2. Cluster Level Competitions or Exhibitions

Cluster level competitions are other learning resources in mathematics. At cluster level, we can have mathematics exhibitions, mental mathematics quiz competitions, mathematics table writing competitions, etc. All these competitions can develop a healthy attitude towards competition, cooperation and coordination among students.

7.5.2.3. e-Learning Laboratory cum Block Resource Centre

e-learning can be understood as learning through electronic means, modes and resources. This may be online or offline, synchronous or asynchronous, etc., but the type, the learner uses, must be an electronic form. In Indian context, if it is not easy to have e-learning laboratories at each school, then as an alternative, we can have e-learning laboratory at block level. While talking about e-learning, Bhatia (2009) mentions that e-learning should be used to supplement and not supplant traditional forms of teaching - learning.

Main features of e-learning are:

(i) connectivity or networking

(ii) flexibility

(iii) interactivity and collaboration

(iv) virtual learning environment like texts, visuals, quizes, etc.

Various e-learning tools can be used for mathematics learning like e-mails, blogs, wikis, e-portfolios, animation, videos, links, specialised softwares, etc. Noss (1988) had a study with 13 years old LOGO experienced children, on and off the computer. Children were asked to solve ratio and proportion problems using computer as well as paper and pencil. He used pencil and paper for ratio test. It was found that the performance was better in case of students solving the problems on the computer. Alongwith these computer related facilities, other learning aids can also be put in block resource centre. Students are suggested to go through various packages produced/being produced by Regional Institute of Education (NCERT), Bhubaneswar and CIET for pedagogy-technology integration discussing various e-tools and their integration in teaching-learning process.

7.5.3 District Level

7.5.3.1 Science Centre

At district level, there are some science centres. These centres inherit mathematics as a science component. There are exhibitions and several other activities at science centres pertaining to mathematics also which may really help in learning of mathematics.

7.5.3.2 DIET

At district level, there are District Institutes of Education and Training (DIET). These institutes have enriched mathematics laboratories which can help prospective teachers to learn more about mathematics and teaching of mathematics, which in turn will help learners. Most of the models and aids in these institutes are being prepared by student-teachers. It can be revealed from NCFTE (2009) that student teachers learn to integrate ideas, experiences and professional skills through hands on experience of developing learning materials.

7.5.4 Open Educational Resources, Web Resources and Virtual Classrooms

Now a days, more emphasis is being given on open educational resources. Since most of the resources on internet are paid resources, it is not possible for all, to access and use these resources for learning. While it is very important that everybody should learn mathematics in the current era, how can we deprive off a major section of the society from new technology and resources for learning of mathematics? The answer to these questions is open learning resources. There are several websites which make available *web content* freely available for all. A very well known name is 'wiki'. Wiki means 'what I know is'. This is a very large project and comprises of various components. The most popular open education resource is Wikipedia. Its website address is http://wikipedia.org. It comprises editable text material and information on almost every topic. If it is not there, anybody can create a page for that particular topic. The information can be seen in almost every language, including English and Hindi. Wikipedia also hosts a number of sister projects which are equally important. Some of these are Commons, Wikiquote, Wikispecies, Wikinews, Wikibooks, Wikiversity, Wiktionary, Wikisource and Meta-wiki. All these resources are open for all anytime, anywhere and that too free of paid services. These sources give freedom to edit and express for every individual. There are Google applications too as other resources.

Some other good web resources in mathematics are:

http://mathforum.org

http://www.algebasics.com

http://www.cutescience.com

http://mathworld.wolfram.com

http://www.ipl.org

http://www.emis.de

http://www.mathmistakes.com

http://www.nctm.org

http://www.awm-math.org

http://www.eric.ed.gov

www.e-book.com.au

www.dli.ernet.in

http://wikieducator.org

There is another resource known as virtual classroom. In virtual classroom, people interacting simultaneously are not face to face, but still there is a sort of synchronous communication among all people. In such a classroom, anybody can express anything anytime during the class and all others will come to know and respond to the query or views expressed. It all happens when people are sitting at their respective places either at home or at likewise setting. So, in such a learning situation, they are not required to assemble altogether at the same place. While talking about importance of virtual classroom, Amin (2010) says "Teachers' physical presence is not needed all the time. Even at higher education level, a person from one country can have their mentors or teachers from other country. This has created a greater impact and given broad outlook to the education." The versatility of such resource is that there can be learners sitting in different continents like Asia, America, Africa and Australia simultaneously and interacting with each other. A website as an example for such classroom can be given as

www.wiziq.com

Students, while using any website, are advised to be cautious about their possible harms too, as some of the websites may hack their computers, misuse their data, transfer virus to their computers. One of the possible solutions/precautions may be that the computer should have been installed with latest updated anti-virus. One of the free anti-virus is 'Clam Win', which may be downloaded from link provided on website of UNESCO (2011). Students are suggested to check the current status for this anti-virus.

7.6 Handling Hurdles in Utilising Resources

While utilising all above mentioned resources, there are some hurdles felt by either teacher or students. A hurdle, in general, can be understood as a difficulty or obstacle that has to be overcome. So if we want to utilise these for better learning of mathematics, we will have to overcome these hurdles. Broadly we can categorise these hurdles into two categories : first, social and ethical hurdles and second, technical hurdles. CALtoonz2006 and Roblyer (2008) talk about various aspects of these hurdles. Let us discuss these in detail.

(a) Social and Ethical Hurdles

As ethics are basically related to the society, so these hurdles can be put together with social hurdles due to their complementary nature. Various hurdles or attention seeking issues with reference to CALtoonz2006 and Roblyer (2008) can be mentioned as:

(i) **Secularity of the Content:** India is a secular country as mentioned in our Constitution. So, we have to maintain secularity in the content of the resource. No resource can be utilised and accepted which harm our secularism.

(ii) **Gender Equity:** Gender equity is another very important issue. Our resource should be based on giving equal importance to both the genders, male as well as female. For the evolution of a modern and developed society, gender equity is important.

(iii) **Democracy:** Democracy is giving equal opportunity and equal rights to all. Our resource content should provide ample instances for reflecting democracy. If it does not reflect, it may not be accepted by the society.

(iv) **Respect for Elders:** The resource should give space for and should inculcate value of respect for elders. Our Indian culture is well known universally for respect for elders. So, if our content of resource reflects respect for elders, it will be heartily accepted by the learners as well as their parents.

(v) **Respect for the Disabled:** This era is the era of inclusion of all in every aspect of our society, including the field of education and betterment of life. Our resource should show equal opportunity to all and it should pave a way giving respect to the disabled.

(vi) **Respect for all the Religions:** The resource content should give respect to all the religions. The content should not have any material which shows disrespect to any religion. This may help in fostering fraternity among learners.

(vii) **Concern for Animals:** Our society is more concerned about animals and their welfare now a days. The resource should respect such concern in its content and presentation.

(viii) Respect for the Environment: While we are stressing for **eco-friendliness** of everything we are using or producing, how can we leave our resource away from such an important issue. Our resource should be eco-friendly, encourage eco-friendliness, hence, should have respect for the environment.

(ix) Plagiarism and Cyber Cheating: Plagiarism is using and mentioning work of some other person without acknowledging that person or mentioning in the name of oneself. This is just like a cheating and unethical that the work of some other person is being used or published in the name of self. So, using any resource in such a way should be avoided.

(x) Illegal Downloads/Software Piracy: Software and media companies are prosecuting offenders of illegal downloading and piracy of softwares. Hence, one should avoid illegal downloading and piracy of softwares.

(b) Technical Hurdles

Some of the technical hurdles are:

(i) Colour: Colour used should not be hot colours. Most of the colours should be soft colours and eye friendly. The learner should not feel more stress while going through the content as well as pictures and figures.

(ii) Speed: In case of multimedia resource, the speed should be optimum enough to provide learners with ample time to go through the content and concept. If speed will be too fast, learner would not be able to go through entire slide and if it will be too slow, learner would feel like wastage of time and it may create disinterest.

(iii) Smoothness of Animation: In case of animation, it should be smooth enough to facilitate learner for better learning. It should not create among the learners a sense of irritation.

(iv) Use of Screen: Entire screen should be efficiently used. It should not be like that the entire content or picture is lying on a corner and majority of the space of the screen is lying vacant or useless. If only text is there, it could have its orientation beginning from centre. In case of books and e-books too, each page can be considered as one screen.

(v) Special Effects: Special effects, if any, should be learner centred. It should be in consideration with the age level, mental level, previous knowledge, attitude and aptitude level and readiness of the learner.

(vi) Music: Music, sound and voices used should be appropriate with respect to validity, timing and relatedness. It should be soft and ear friendly. Under no circumstances it should be harmful or irritating to the learner.

If all these social, ethical and technical hurdles can be overcome, then not only learners and teachers, but everybody related to the field of mathematics will use these resources for betterment of learning and hence, for the betterment of the entire field of mathematics.

Some suggestions for overcoming the hurdles could be as follows:

(i) **Narration:** It can be better, if the text for using at school is in the form of narration. A narrator should always be present over there. It means, if a text is being represented, it should be shown in such a way that it is being narrated by some character instead of simply writing the text in open space.

(ii) **Teacher Friendliness:** It should be easy for the teacher to handle the resource. If teacher, using the resource will not feel comfortable, it may create disruption from using the resource in future again.

(iii) **Teacher Training and Skill Development:** A teacher should be trained and provided with ample skills to use resources in learning of mathematics. It must be a compulsory part of a teacher training programme, that prospective teachers be given training for skills to handle learning resources.

(iv) **Attitude and Ease of Access:** The source should be easily accessible to all the students and teachers. Though they may not have a positive attitude towards utilising these resources in learning and teaching, but ease of access will surely motivate them for utilising these resources in learning, 'teaching of mathematics.'

If all these social, ethical and technical hurdles can be taken care, then not only learners and teachers, but everybody related to field of study of mathematics will use these resources betterment of learning and hence, for the betterment of the entire field of mathematics.

EXERCISE 7.1

1. Critically analyse the role of textbooks and supplementary books in learning of mathematics. Elaborate your answer.
2. Why the selection of appropriate multimedia is important? What are the possible threats which may arise, if an inappropriate multimedia is selected?
3. Discuss various steps for designing multimedia in mathematics in detail by taking an appropriate example.
4. Explore the availability of learning resources for mathematics, if you go for a community excursion for a week in a village.
5. In the capacity of mathematics teacher, what steps would you take for pooling of learning resources at school level? How these resources can further help in pooling of resources at district level?
6. Critically examine the role of ethical hurdles which may come across while using resources in mathematics learning. How can you overcome technical hurdles too in this regard?

Summary

Learning resources play a vital role in learning of mathematics. Here and there, are spread several resources which can be used for learning of mathematics. These can be in the form of textbooks, hand books, reference books, supplementary books, audio-visual multimedia or community resources. Sometimes, there is a need of pooling these resources at formal and informal levels. Various resources, mentioned here, provide powerful vehicles to engage and sustain children's interest in mathematics. If these resources are properly used as learning opportunities for mathematics learners, it will surely empower our learners to think like mathematicians.

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UNIT 8

ASSESSMENT AND EVALUATION

Jatin, a Class IX student is always anxious about mathematics examination. This time, he practised each type of question at least five times and got a score of 32 out of 50. He is relieved, by the feeling that he managed to pass.

Chaitanya, a Class VIII student enjoys doing mathematics and helps his friends also in solving problems in different ways. He knows many tricks of solving problems and loves to play with numbers. He got 87 out of 100 in mathematics examination as he did not write the 'desired steps' for solving problems.

Jaspreet always scores above 90% in all mathematics tests. But, she does not enjoy solving mathematical puzzles.

What do you infer from these narratives?

Who is 'good' in mathematics? Are the tests/examinations measuring the learners' 'aptitude' in mathematics ? What do examination marks/scores indicate? What are your ideas about evaluation and assessment based on from your personal experience?.

8.1 Introduction

Assessment and evaluation are integral components of teaching-learning process. They not only provide feedback about learners, but also about the effectiveness of curriculum, programmes and policies. We often use the terms–Assessment and Evaluation interchangeably, but it is important for us to distinguish between them. The meanings and scope of these terms, as applied to educational setup, are explained below:

Assessment is defined as the process of obtaining and documenting the information about the subject, skills, attitudes and beliefs of the learners. It is an interactive process between students and teachers that informs teachers about the effectiveness of their teaching and the level of students' understanding/learning. Assessment provides feedback for the purpose of evaluation of learning outcomes and future performance. When we say that we are "assessing a student's competence," we mean that we are collecting information to help us decide the degree to which the student has achieved the learning targets. A large number of assessment techniques–formal and informal observations of students, paper-pencil tests, projects, assignments, etc. can be used to gather information.

Evaluation is defined as the process of making a 'value judgement', i.e., assessment about the worth of a student's performance. So, through assessment of the student, the evaluation is carried out.

The distinction between assessment and evaluation can be understood by following example:

A teacher has to identify students writing ability, so that they can take part in a National level essay competition. In this case, the teacher has to make a value judgement about the writing skills of the students. For this, the teacher has to 'assess' students' writing abilities by gathering information from students' previously written essays etc. and compare with other students as well as with known quality standards of writing. Such assessment strategies provide information which can be used to judge the quality or worth of the student's writing.

The purpose of assessment is depicted in Fig. 8.1*

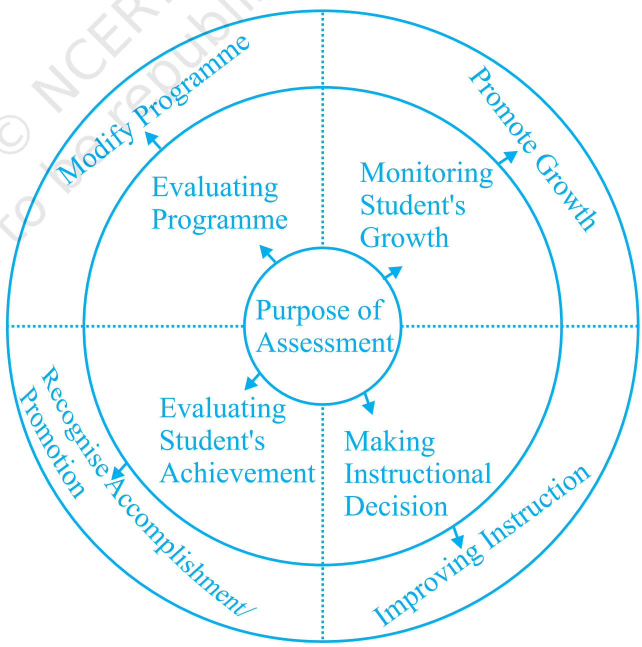


Fig. 8.1

* Source – Assessment Standards for School Mathematics NCTM(Reference-www.nctm.org)

An assessment is authentic, if the assessment procedures match with what children are learning. It provides them with the feedback about their progress in mastering new knowledge. Authentic assessment acknowledges that learners learn differently and hence should get opportunity to express their learning in multiple ways. This Unit describes various assessment strategies which the teacher can use for formal and informal evaluation.

Learning Objectives

After studying this Unit, student-teachers will be able to:

- understand the role of informal evaluation in mathematics
- know various assessment strategies which can be used for informal evaluation
- understand the role of formal evaluation in mathematics
- know various assessment strategies which can be used for formal evaluation in mathematics
- know the merits and demerits of various assessment strategies
- understand the meaning and scope of continuous and comprehensive evaluation
- develop an understanding of assessment framework
- construct variety of questions and a question paper.

8.2 Informal Creative Evaluation

Learning is a continuous process and children 'construct' their own knowledge in multiple ways. Classroom interaction provides a wide range of opportunities to make observations of a learner's change in behaviour and learning. As we are aware, some of the observations are made on a daily basis in an 'informal' manner while teaching-learning is going on. These observations give us an idea not only of the learner's strengths and weakness, but also of the other aspects like attitude towards the subject, problem-solving abilities, creativity, divergent thinking, closeness to the subject, psychomotor skills, etc. In this way, 'informal evaluation' helps the teacher to assess the learners in multiple ways and provides a holistic understanding of the learner. However, these day-to-day or informal ways of observations may easily be forgotten, if not recorded in a systematic manner. This recording should include records of observations, comments, rating of the learner's performance, anecdotes and incidents indicating their attitudes and dispositions towards the subject area etc.

Informal creative evaluation in mathematics includes assessing the following:

- Mathematical communication
- Understanding of mathematical concepts and processes

- Creativity in mathematics
- Problem-solving abilities and mathematical reasoning
- Disposition and attitude towards mathematics

In order to assess the above mentioned areas, a variety of methods or tasks can be used by the teacher which are discussed in the forthcoming section.

'Informal' and 'creative' evaluation in mathematics not only provides scope to assess the learner in multiple ways, but provides a stress free and learner friendly environment in the classroom. It reduces the anxiety of non-performance as in the case of formal examination. Informal evaluation also helps the teacher to identify the gifted learners in the classroom. It aims to create varied contexts and opportunities for learners to perform in accordance to their learning needs.

8.2.1 Methods of Assessment in Mathematics

There are variety of assessment methods and tasks which can be used by the teacher in an 'informal' and 'creative' way so as to understand the learner in an holistic manner.

Some of these are discussed in the following sections:

8.2.1.1 Anecdotal Records

Anecdotal records are written records/descriptions about the learner maintained by the teacher which generally includes observations and narratives. They not only help in identifying strengths and weaknesses of the learner, but also his progress over a period of time.

8.2.1.2 Checklists

Checklist mentions presence or absence of student's behaviour (cognitive as well as affective) expected during daily performance. The items on the checklist may be related to behavioural or content area objectives. A checklist, as an observational technique, can document the degree of the learner's accomplishment of competencies within mathematics curriculum.

An example of checklist is given below:

Name of the Student__________

Mathematics Area	Competencies	Yes/No
Problem-solving	Did the learner • understand the problem? • use more than one strategy to solve?	
Reasoning	Did the learner • recognise patterns? • make prediction? • justify the solution?	

8.2.1.3 Rating Scales

Rating scales are extended form of checklists. In rating scales, we create standards or criteria for evaluating a performance and each standard has a definite level of competence and we rate learners according to how well they perform on each standard as they complete the task.

Example of Rating Scale

Student shows interest in solving problems:

5	4	3	2	1
Very high	High	Moderate	Low	Very low

8.2.1.4 Journals Writing

Journal writing helps the learners to reflect on their learning in the classroom and provide opportunity to express their opinion. This process of writing about learning encourages learners to analyse what and how they have learnt in the classroom. They provide teachers with insights into the ways in which students have constructed knowledge and into their levels of conceptual understanding.

8.2.1.5 Rubrics

A 'rubric' provides written guidelines by which student's work is assessed. The term rubric is defined as

"a set of guidelines for assessment which states the characteristics and/or dimensions being assessed with clear performance criteria and a rating scale."

A scoring rubric consists of

- A fixed scale
- A list of characteristics describing performance for each of the points on the scale
- A clear performance target for students.

Rubrics are useful for both learners and teachers as they make evaluation less subjective.

Example of a Mathematics Rubric for a problem-solving task:

Level 4	Level 3	Level 2	Level 1
• shows complete understanding of mathematical problems, concepts and principles	• shows nearly complete understanding of problems, concepts, etc.	• shows partial understanding of the problems concepts	• shows no understanding of the problem. Needs revisiting the concepts
• uses appropriate mathematical terminology and notations	• uses nearly correct terminology and notations	• uses incorrect notations	• does not use any notations
• executes algorithms completely and correctly	• executes correct algorithms, but computation has minor errors	• executes incorrect algorithm	• no algorithm is executed

EXERCISE 8.1

1. Explain how informal evaluation helps in assessing mathematical understanding of learners.
2. How can you incorporate observational assessments into your daily lessons? Discuss one method of getting observations recorded.
3. If you have to choose an assessment strategy to profile your learners, which one would you choose and why?

4. Describe the essential features of rubric, and explain how students can be involved in understanding and using rubrics to help in their learning.
5. Choose a problem-solving task and develop a rubric to assess your learners.
6. What are the advantages of checklists and rating scales?

8.2.2 Assessing Creativity and Problem-Solving

Creativity in mathematics is characterised by (i) divergent thinking in problem-solving, (ii) problem posing and (iii) redefining (where the pupil is required repeatedly to redefine the elements of a situation in terms of their mathematical attribute).

Some of the indicators of creativity are:

- the learners use multiple and alternate problem-solving strategies
- the learners are able to think flexibly
- the learners can develop their own heuristics to solve problems
- the learners use divergent thinking in different contexts and situations
- the learners can reverse their direction of thought
- the learners can create generalised mathematical relationships
- the learners can pose questions or create problems from the given data.
- the learners can create unusual inter-relation among mathematical concepts.

For assessing the above mentioned attributes or characteristics, our assessment tasks should be such that they are able to identify the learners having such attributes. The routine tests and drill exercises in mathematics seldom give such opportunity. So, the teacher need to give such situations in the classroom by creating appropriate tasks. Open-ended questions, dialogues on problem-solving and reflecting on own problem-solving strategies are some of the ways to assess creativity.

8.2.3 Assessing Experimental Work in Mathematics

'Experimentation' in mathematics and 'Mathematics Laboratory' have become integral parts of learning strategy for school mathematics. It is important for us to devise well thought out activities and strategies and assess them in laboratory mode. Mathematical experiments and laboratory activities focus more on 'hands on learning' in mathematics, and hence, various process skills like observation, measurement, data collection, experimentation, etc. need to be assessed. For this purpose, the written records of the students can also be assessed.

Some criteria for assessing laboratory activities in mathematics through checklist are:

- Does the learner have sound understanding of mathematical concepts?
- Has the learner worked with precision and neatness?
- Has the learner been able to interpret the data?
- Has the learner been able to draw valid conclusions?

These criteria could be translated into a rubric or checklist or rating scale

Some examples of experiments in mathematics are illustrated as follows:

Examples of Experiments in Mathematics:

- Verification of various properties of triangles.
- Generating patterns using Pascal's triangle
- Construction of Pythagorean triplets.
- To verify that the given sequence is an arithmetic progression.

For further details, refer to: "Designing of Mathematics Laboratory in Schools and Laboratory Manuals in Mathematics," published by NCERT.

8.2.4 Self and Peer Evaluation

Self evaluation refers to the student's own evaluation of his/her learning and progress in knowledge, skills, processes, interests, attitudes, etc. Self evaluation helps the students to develop a sense of independence and take responsibility for their own learning. In a constructivist classroom, this aspect has a lot of importance as it makes the learners accountable for their own learning. Self evaluation also helps the learners to identify their own strengths and weaknesses and monitor their own progress over a span of time. This type of assessment is very reflective.

Peer evaluation refers to evaluation of a student by another. This can be conducted in pairs or in groups. Peer assessment helps the students to work collaboratively and share ideas. This promotes cooperative learning environment in the classroom. This also helps the students to develop a deeper understanding of a topic or concept. The student also gets opportunity to share and exchange work with others.

Peer evaluation can be done using checklists, rating scales, rubrics or by qualitative feedback.

To develop an effective scheme of peer and self evaluation, the learning objectives and criteria of assessment should be explicit and transparent to the students.

EXERCISE 8.2

1. What do you understand by creativity in mathematics? Explain with examples how you can assess creativity in mathematics.
2. Design any five laboratory activities for secondary school students to assess process skills in mathematics.

3. What are the advantages of self and peer evaluation? Do you think any disadvantages of self and peer evaluation? Explain.
4. What is the importance of experimentation in mathematics? Also, explain how a teacher can assess students performance on experiments?

8.3 Formal Ways of Evaluation

Formal evaluation of learners helps the teacher in many ways, such as assigning grades, placement of students, promotion to the next level, effectiveness of teaching, assessing the learners conceptual and procedural understanding, diagnose learners' errors, judging the effectiveness of a programme, curriculum, etc. Formal evaluation involves systematic gathering of information and interpreting it. Formal evaluation as compared to informal evaluation is more objective, pre-planned and especially suitable for large groups.

Depending on the purpose, formal evaluation, can be of following types:

1. **Formative Evaluation –** Formative evaluation refers to the 'ongoing' and systematic assessment of student's achievement while the instructional programme is in progress. Feedback to the students is the main purpose of formative evaluation. Thus, it is a continuous and integral part of teaching-learning process in the classroom. Formative evaluation helps both the students and teachers in assessing the conceptual and procedural understanding.
2. **Summative Evaluation –** The summative evaluation refers to final evaluation at the end of a term, course or instructional programme. It involves decision-making like placement' assigning grades/ranks to the learner. It is a means to an end whereby the focus is to determine the present status or position of a student in that subject area.
3. **Diagnostic Evaluation –** Diagnostic evaluation as the name indicates refers to the evaluation procedure whereby we can identify the nature and degree of learning difficulties in the students. Diagnostic evaluation involves gathering information about the students errors, reasons for those errors, depth of conceptual understanding, effective means of intervention or remediation.
4. **Prognostic Evaluation –** Prognostic evaluation helps in predicting the probable degree of success in a particular subject area. They test the background skills and abilities which are prerequisite for success in that particular subject area, namely, aptitude tests.

Formal evaluation can be done using a variety of assessment techniques and practices which are discussed in the following section:

8.3.1 Assessment Techniques and Practices

Apart from paper-pencil or written tests, there are a variety of assessment techniques which can be used for formal evaluation. Also, the methods discussed for informal evaluation can be used for the purpose for formal evaluation.

8.3.1.1 Written Tests and Tasks

Written tests are often used as *Achievement test.* Achievement test is an instrument designed to measure the accomplishment of the students, in a specified area of learning, after a period of instruction. Hence, written tests developed for the purpose of testing the achievement of the students can be given at the end of Unit, term, semester, year, etc.

Written tasks are usually referred to as problem-solving exercises which are given to the students to assess their conceptual and procedural understanding.

Depending upon the type of interpretation used, the written tests may be:

- Norm referenced
- Criterion referenced

Tests which provide information about the relative performance of members of a specific group of students, are called 'norm referenced tests'. In norm referencing, the interpretation of the test scores is done by comparing them to scores obtained by other students of the norming group. The scores obtained through a norm reference test would be classified into different categories like average, above average, below average, etc. Norm referenced tests are helpful for relative comparison. In criterion referenced tests, the written test is based on a set of predetermined objectives and provides information about students' degree of proficiency in the achievement of objectives of which the test is a measure. Hence, the scores help us in judging the achievement of objectives.

Depending on 'who' prepares the tests, the written tests may be:

- Teacher made
- Standardised

Teacher made tests are usually criterion reference tests which are developed by the teacher to measure the student's achievement against predetermined criteria. Standardised tests are usually norm referenced tests and are constructed for large group of students. Depending on the type of questions, the written tests/tasks in mathematics may be:

- Subjective/Descriptive
- Objective

The subjective test in mathematics usually contains essay type questions and short answer type questions, that is the questions for which procedure of solution is also evaluated. The objective type tests contain objective questions like multiple choice questions, fill in the blanks, one word answers, etc. These questions are also called test items. The details of these types of questions, and of how to construct them are discussed in sub unit– 'Assessment Framework' in the forthcoming sections. Subjective tests are usually preferred when we want to evaluate procedural as well as conceptual knowledge, whereas objective tests are used to evaluate conceptual knowledge.

Advantages of written tests:

- They are very important tools for summative evaluation
- They help in determining the relative position of a student in the subject
- They provide the teacher with evidence relating to the attainment of objectives, effectiveness of learning experience of the students
- They help the teacher to identify the student's difficulties, gaps in conceptual and procedural knowledge and hence, helpful in taking remedial actions
- They are useful in giving feedback to parents, administrators, etc.

8.3.1.2 Assignments

Assignments can be class assignments or home assignments. They could also be individual or group assignments. Written assignments are helpful to the learner in planning, composing and working out in a particular unit/theme. Assignments help to assess a wide range of objectives and content of learning and also provide the learners an opportunity to relate and synthesise within and outside classroom learning. Assignments also encourage creativity as it provides scope to the learners to articulate their ideas in multiple ways and through written and visual expressions. Teacher can desgin specific assignments for diverse ability learners in the class.

Example:

- An assignment in statistics could be given to collect data from newspapers on a particular topic and represent them using Bar graph and Pie diagram.
- An assignment could be given to frame 5 questions from the given data in statistics.

8.3.1.3 Projects

A project is a motivated problem, solution of which requires thought and collection of data and its completion results in the production of something of value to the students.

Project enables learners to conduct real inquiry in an interdisciplinary manner. It promotes problem-solving in mathematics and connects it to real life application.

Projects in mathematics provide opportunity to observe, collect data, analyse, organise and interpret data and draw generalisation.

A project could be individual or group project and could be presented in the form of a document, report and/or a multimedia presentation.

Examples of Project Level-VIII to X

- **Geometry in Real Life:** This project can enable the students to apply geometrical concepts, such as properties of triangles in real-life situations. Students can find the height of buildings, trees, etc.
- **Project on BMI (Body Mass Index):** In this project, students can investigate health conditions of a sample population by calculating B.M.I. The detailed surveys, calculations, graphs, tables, etc. can be used to depict the results of the project and also this project draws an interdisciplinary linkage with biology.

The criteria of assessment of the project could be translated into a well defined rubric.

Example

Criteria	Level 4	Level 3	Level 2	Level 1
Content	Accurate, precise, relevant and interesting	Accurate, precise, but not so interesting	Content has some errors, relevant but not so interesting	Content is accurate and not relevant
Creativity	Very high	High	Moderate	Low
Organisation	Very well organised and sequenced	Well organised and sequenced	Not so well organised	Not well organised and content is not sequenced
Originality	The information is well researched and original	The information is well researched	The information is not original	The information is completely copied

8.3.1.4 Portfolios

Portfolio is a collection of evidence of a person's skills. For the purpose of assessment, a portfolio is a limited collection of student's work which gives evidence of learners' meaningful learning and demonstrates their growth over a span of time.

Advantages of Portfolio

- Promotes problem-solving and thematic approach.
- Shows growth in acquiring knowledge and skills.
- Encourages students to collect, organise and reflect on their own learning.
- Provides scope for multiple means of knowledge representation.
- Encourages performance in non-traditional and non-language dependent medium.

For these reasons, portfolios are increasingly becoming popular as authentic means of assessment.

Content of a Mathematics Portfolio

A variety of items can be included in a portfolio which allows the students to work over a span of time.

Some of the suggested items are

- open ended questions
- a report of group project
- problems posed by the students
- a small project
- a book review
- experts from students' journal
- newspaper and magazine articles
- a mathematical research
- problem-solving tasks
- self evaluation, etc.
- peer evaluation, etc.

8.3.2 Assessing Product vs Process

Process and product are intimately related, although it is easier to assess product than the process. However, in mathematics both process and product play important roles. That is the reason always weightage or marks are allotted to all steps. Descriptive tests and tasks often assess both process and product, whereas in objective tests more emphasis is laid on product evaluation. However, it is important to note that we seldom reach correct answer (product), unless we employ correct process.

Assessing the process entails assessment of conceptual as well as procedural knowledge in mathematics. Often we come across students, who employ correct process or algorithm, but may get wrong answer due to computational error. Employing a correct process requires deep understanding of algorithms. The laboratory activities and experiments in mathematics are so designed as to foster process skills. Some important process skills in mathematics are estimation, drawing, observation, generalisation, classification, interpretation, hypothesising, etc.

Therefore, our assessment techniques should be so designed that we can assess these process skills in mathematics.

8.3.3 Mid-term and Terminal Examinations

Mid-term/terminal examinations are basically a part of summative evaluation. They help in assessing the composite achievement of the learner during that time span. These examinations constitute the formal evaluation.

The mid-term/terminal examination provides a basis of promotion to the next higher grade. They also help in determining the relative position of a student in that subject.

Mid-term/terminal examinations are usually written examination. The question paper for this examination needs to be crafted carefully according to the weightage of each Unit and objectives.

It can include variety of questions like multiple choice, essay type, story problems, etc. This is discussed in detail in Sub Unit assessment framework in the forthcoming section.

8.3.4 Continuous and Comprehensive Evaluation

Following the recommendations of NCF–2005 on evaluation reforms in school education, the traditional examinations are being replaced with continuous and comprehensive evaluation (C.C.E.) scheme.

The term 'continuous' is meant to emphasise that evaluation of identified aspects of students' growth and development is a continuous process rather than an event, built into the total teaching-learning process and spread over entire span of academic session. 'Comprehensive' means that the scheme of evaluation attempts to cover both the scholastic and coscholastic aspects of student's educational growth and development.

C.C.E. represents a paradigm shift from rote learning to participatory learning, from a single examination system to alternate and multiple assessment system.

This scheme of evaluation is providing a 'legitimate space' for more flexible and authentic assessment strategies. Hence, under this scheme of evaluation, projects, portfolios, groupwork, etc. are part of formal evaluations.

EXERCISE 8.3

1. Differentiate between formative and summative evaluations and explain their role in mathematics assessment.
2. What is the importance of 'Projects' in mathematics? Explain with examples how you will evaluate projects in mathematics.
3. What are the advantages of 'Portfolios'? Choose a topic and specify what tasks could be included in the portfolio?
4. Which is more important to assess in mathematics-process or product? Justify.
5. What do you understand by continuous and comprehensive evaluation? Explain with suitable examples what evaluation strategies could be incorporated in formative evaluation and summative evaluation.

8.4 Assessment Framework

An assessment framework is a plan or blueprint which helps us in assessing various cognitive and affective learning targets and themes. This establishes a comprehensive and consistent approach for gathering and using information and interpreting it.

8.4.1 Developing a Framework for Question Paper

An assessment framework or blueprint describes both the content and level of performance expected of the students in relation to that content. Thus, the framework includes:

- Content - topics and subtopics to assess
- Levels of cognitive processes
- specific learning objectives
- emphasis for each learning target to be assessed.

The body of the blue print lists the specific learning targets which are indicated by content as well as level of complexity in terms of the taxonomic category. Hence, a systematic and stepwise procedure is required to organise components for developing a framework of question paper. These are discussed as follows:

(i) Identification of the Objectives and Allotting Weightage

The most important step while planning a test or any evaluation tool is the identification of the instructional objectives and stating them in terms of specific observable behaviour.

Example

Content	Objectives	Specific Behaviour	Marks Allotted	Percentage
1.	Remembering	ability to define, state, recall formula, etc.	10	20%
2.	Understanding	ability to interpret, estimate, communicate, explain, calculate, etc.	10	20%
3.	Applying	ability to apply, construct, relate, solve, etc.	15	30%
4.	Analysing	ability to reason, deduce, examine, etc.	5	10%
5.	Evaluating	ability to compile, design, formulate, etc.	5	10%
6.	Creating	ability to conclude, judge, discriminate, etc.	5	10%
			50	100%

This illustrates classification of objectives according to Bloom's revised taxonomy. However, other taxonomies could also be used.

(ii) Selection of the Content and Assigning Weightage

Content is the means through which objectives are attained. Hence, it is necessary to decide the weightage for each topic. When this is done, a decision about the weightage to be given to these units has to be taken so as to represent the actual emphasis laid on them in instruction.

The example is illustrated.

Class IX

S. No.	Unit	Marks	Percentage
1.	Polynomials	10	20%
2.	Probability	8	16%
3.	Quadrilaterals	12	24%
4.	Triangles	20	40%
		50	100%

(iii) Form of Questions and Their Weightage

The teacher or test maker has to decide about the form of questions to be used, the number of questions to be chosen and relative weightage to be given to each form of judicious combinations of the different forms for developing an achievement test.

The table below shows weightage to different forms of questions:

S. No.	Form	Marks	Percentage
1.	Essay (E)	6	24%
2.	Short Answer (SA)	9	36%
3.	Objective Type (O)	10	40%
	Total	25	100%

(iv) Distribution of Difficulty Levels

A decision also has to be taken concerning the distribution of difficulty level. The distribution of difficulty level in a test will depend upon the purpose of the test as also on the group of students for whom it is designed. To have optimum discrimination through a test, most of its questions should be of average difficulty level. If the test is assumed to be normally distributed, some weightages in terms of percentages are suggested as follows:

S. No.	Difficulty Level	Marks	Percentage
1.	Difficult Questions	8	16%
2.	Average Questions	34	68%
3.	Easy Questions	8	16%
	Total	50	100%

(v) Preparation of Blue Print

A blue print is a three-dimensional chart showing the weightage given to the objectives, content and the form of questions in terms of marks. It is called a table of specifications or an assessment framework. The blue print defines as clearly as possible, the scope and emphasis of the test and relates objectives to the content. Once the blueprint is ready, the test items or questions are prepared.

Example of a Blueprint :

Objective / Type of Q's / Content	R	U	A	AN	EV	C	Total
	E SA O	E SA O	E SA O	E SA O	E SA O	E SA O	
Polynomials	2(2)		3(1)	5(1)			10(4)
Probability		2(2)	3(1)			3(1)	8(4)
Quadratic Equations	2(1) 2(2)	1(1)	5(2)			2(1)	12(7)
Triangles	5(5)	5(2)	5(1)	5(1)			20(9)
Total	11(10)	8(5)	16(5)	10(2)		5(2)	50(24)

Note: Number outside the bracket indicates the total marks and inside indicate number of questions.

For example 5(2) means-2 questions of 5 marks.

R – Remembering

U – Understanding

A – Applying

AN – Analysing

EV – Evaluating

C – Creating

Preparing a Question Paper

While designing a question paper, some basic considerations are to be kept in mind. First of all, the assessment should focus on the important learning targets. The question paper should be such that it elicits from students, only that information which is relevant to the learning target being assessed. However, the questions are sometimes made so poorly, that they elicit unwanted behaviours from students, such as bluffing, fear, wild guessing, craftiness of testwise skills. (Testwiseness is the ability to use assessment strategic and clues from poorly written items).

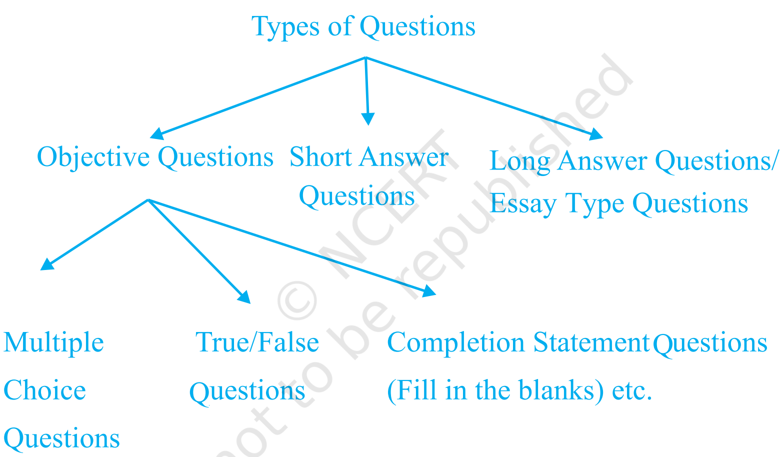
Another important thing is that question paper should neither prevent nor inhibit a student's ability to demonstrate that he has achieved the learning targets. Improvised wording in a question paper may make an item so ambiguous that a student who has the correct knowledge may answer it wrongly. Simple matters, such as vocabulary, poorly worded directions, poorly drawn diagrams may lead students to respond incorrectly.

A question paper should have balance of all types of questions. The types of questions and framing of questions are discussed in the next section.

At the end of preparation of question paper, item analysis should be done in terms of the following information about each item (question):

- Content
- Objectives
- Difficulty level
- Form or type of questions

8.4.2 Framing Questions



Questions should be framed in such a way so as to encourage critical thinking, promote logical reasoning and discourage mechanical manipulation. As questions play a very important role in summative assessment, framing questions requires careful deliberations and should be in alignment with the learning targets. Questions needs to be framed so as to assess not only the concept, but also the skills. There should be balance between conceptual knowledge questions and procedural knowledge questions.

Each type of Questions along with examples are discussed in the following section:

Essay Questions

Essay type/long answer type questions in mathematics are usually problems or questions in which conceptual and procedural knowledge is tested. Higher mental process, such as application, analysis, synthesis, problem-solving, etc. can be easily assessed by essay type questions. These types of questions require step by step procedures and in such questions the procedural knowledge and algorithmic knowledge plays an important role. The problem-solving approach is assessed rather than just the solution. In such type of questions, learner's hypothetic educative thinking, inductive and deductive reasoning and other areas like use of axiomatic knowledge, manipulating the mathematical equations, translating the word problems into mathematical statements are assessed.

Example

1. Prove that bisectors of angles of a parallelogram form a rectangle.
2. State and prove Pythagoras theorem.
3. A hemispherical dome of a building needs to be painted. If the circumference of the base of the dome is 17.6 m, find the cost of painting it, given the cost of painting as Rs 5 per 100 cm^2.

Multiple Choice Questions

A multiple choice question has alternate choices, options or responses amongst which usually one is correct and is known as key. The remaining incorrect alternatives are called as distractors.

The multiple choice questions can be used to assess a greater variety of learning targets and the distractor which a student may choose may give us diagnostic insight into the difficulty the student is experiencing. It also helps to know the misconceptions of the students.

However, multiple choice questions should be crafted so as to minimise the 'chance' or 'bluff' answers. Multiple choice questions in mathematics are more suited to test conceptual knowledge rather than to assess algorithmic or procedural knowledge. Also, multiple choice questions are more popular with learners as they do not require to write step by step procedures.

Examples of Multiple Choice Questions

Example 1. The total surface area of a cylinder is

(a) $2\pi r(r + h)$

(b) $2\pi rl$

(c) $2\pi r^2$

(d) $2\pi r^2(r + h)$

Example 2. Meena has to prepare a model of a cylindrical kaleidoscope for her science project. She wants to use chart paper to make the curved surface of the kaleidoscope. What should be the area of chart paper required, if the length of kaleidoscope is 2.5 cm and radius is 3.5 cm?

(a) 550 m^2

(b) 550 cm^2

(c) 1050 cm^2

(d) 1100 cm^2

True/False Questions

True/False test items are limited option multiple choice questions. These test items are often used to test the conceptual knowledge in mathematics.

Example: There are infinite number of lines which pass through two distinct points (True/False)

True/False test items could be improved by asking the learners to justify their response or ask them to correct the false statement and also could be followed by reasoning.

Statement Completion Questions

The completion variety requires a student to add words to complete an incomplete statement. They are mainly used to assess students' performance of lower order thinking skills, such as recall and comprehension of information. However, in some cases they can also be used to assess higher order thinking skills like ability to manipulate mathematical symbols and balance mathematical equations.

Example:

- A quadrilateral is a parallelogram, if a pair of _______ sides is equal and parallel.
- If $(x - 1)$ is a factor of $4x^3 + 3x^2 - 4x + k$, then value of k = _______.

8.4.3 Framing Open Ended Questions

Open ended questions in mathematics promote divergent thinking, creativity and higher order thinking skills in the learner. These are also called divergent questions. Open ended questions have a scope of multiple answers and give opportunity to the students to express their ideas in a number of ways. These questions have alternate approaches and give learners scope to answer in their own words. This discourages rote learning and unnecessary memorisation and helps in cultivating creativity. Through such questions, the teachers can assess the originality of the students. This also helps the teacher to assess the conceptual gaps in the learner.

Framing open ended questions requires careful deliberation on the part of the teacher. They should be framed such that learners find them interesting and challenging. Open ended questions help both in formal and informal evaluation.

Example: Closed ended (convergent) question :

Prove Pythagoras theorem.

On the same topic open ended question like the following could be framed:

Q. 1. Think of two ways to verify Pythagoras theorem using concrete material.

Other examples of open ended questions:

Q. 2. Find out as many rectangles as possible which have area of 64 m^2.

Q. 3. Find three rational numbers between two given rational numbers.

Q. 4. Find three irrational numbers between two rational numbers

8.4.4 Framing Conceptual Questions

Questions framed should be such that they assess the conceptual knowledge of the learners. These give feedback to the teacher about how well students have learnt concepts.

Some Examples

Instead of asking the students to list the properties/operations of real numbers, question can be framed as:

Q. Which of the following operation is not defined for the real numbers?

(a) $3 + 0$ (b) $\frac{3}{0}$ (c) 3×0 (d) $0 - 3$ (e) $\frac{0}{3}$

Q. Which of the following cannot be the probability of an event?

(a) $\frac{1}{3}$ (b) 10% (c) 0.42 (d) $-\frac{1}{2}$

Q. Instead of asking the students to bisect a given line segment, using a ruler and compass the question can be reframed as follows:

In bisecting a line segment, using a ruler and compass, you

(a) also construct a perpendicular to the line segment.

(b) use a pencil, protractor and set squares.

(c) need an accurate ruler for measuring.

(d) none of these.

Such questions help the learners to grasp the conceptual understanding and improve their problem-solving skills and divergent thinking.

Some more examples of conceptual questions:

Q.1. A transversal intersects two lines in such a way that two interior angles on the same side of the transversal are equal. Will the two lines always be parallel?

Q. 2. The mean of ungrouped data and the mean calculated when the same data is grouped are always the same. Do you agree with this statement? Give reason for your answer?

Q.3. Circumferences of two circles are equal. Is it necessary that their areas are also equal? Is the converse also true?

For more questions, you can refer to *Exemplar Problems in Mathematics* for Classes IX to XII published by NCERT.

EXERCISE 8.4

1. Explain various steps involved in developing a question paper.
2. What is the importance of making blue print?
3. Explain what precautions should be taken while making a balanced question paper.
4. Discuss the advantages and disadvantages of multiple choice questions in mathematics.
5. Choose any topic and frame five open ended questions. Mention the learning objectives which are likely to be attained.
6. Choose a topic for Class IX and frame five conceptual questions and explain how they are different from procedural questions.

Summary

In this Unit, we have discussed the purpose and methods of assessing student's mathematical learning. It is clear that monitoring students' progress and discussing understanding of achievement are critical for student's learning and as mathematics teachers, we can use many different methods to gather information about student's understanding and skills. While assessing children, it is important to appreciate differences amongst them and respect the fact that they will understand and respond in different ways while learning. Assessment of content as well as skills are crucial for effective mathematics instruction. The various assessment strategies and tools discussed in this Unit, empowers the teachers to make assessment in mathematics more meaningful, authentic multidimensional and stress free.

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UNIT 9

MATHEMATICS FOR ALL

9.1 Introduction

This is no denying fact that mathematics, in one form or the other, is essentially needed in every walk of life– home, business, industry, sale-purchase, banking, agriculture, communication, transport, defence, science and technology, etc. In view of this, learning of mathematics at school stage becomes more significant. So, children's activities at school must also be linked to their lives outside school. For this purpose, cooperative learning, creativity, inventiveness, laboratory approach teaching and recreations in mathematics may act as a catalyst in mathematics learning for all. Teachers are expected to identify learner's strengths and weaknesses and undertake activities enriching mathematics learning so as to help children mathematise the World around them. Therefore, personal qualities and character, educational qualifications and professional competence of a teacher is utmost important on which the success of teaching - learning programme depends.

In this Unit, activities enriching mathematics learning for all, such as supplementary text materials including use of mathematics laboratory, summer schools, correspondence courses, mathematics clubs, cooperative learning, stimulating creativity and inventiveness and some recreational activities have been discussed.

Learning Objectives

After studying this Unit, the student - teachers will be able to:

- Identify learner's strengths and weaknesses in mathematics.

- Undertake the following activities for enriching mathematics learning:
  - — Assisted learning
  - — Use supplementary learning materials, including mathematics laboratory
  - — Organise summer programmes
  - — Use distance education techniques for continuing education
  - — Establish mathematics clubs
  - — Organise contests and fairs
  - — Create recreation in mathematics through games, puzzles, riddles, etc.
  - — Develop understandings in mathematics through peer cooperation
  - — Stimulate creativity and inventiveness in mathematics.

9.2 Identifying Learners Strengths and Weaknesses

9.2.1 Strengths and Weaknesses of Learners

The teacher is an independent factor who plans, organises, leads and controls his teaching. He is free to perform various activities as per the needs of the learner, who being a dependent factor, is required to act according to the planning and organisation of the teacher. Thus, it becomes the responsibility of the teacher to know the strengths and weaknesses of the learner and diagnose accordingly.

Strengths of learners must be channelised and weaknesses need to be *addressed.* These weaknesses are usually of two types:

(i) **General Weakness:** This means alround weakness. It is almost inherited and is due to a low intelligent quotient (I.Q.). The removal of this weakness is a joint responsibility of all the teachers in the school. Of course, the mathematics teacher is required to pay special attention in the case of his subject.

(ii) **Particular Weakness:** This is a case which has weakness in mathematics, but fairly strong in other subjects. Removal of this weakness is easier than removal of general weakness. This weakness is the sole concern of mathematics teacher. Acute cases may be treated in separate classes whereas border line cases may be tackled in the classroom itself.

In a heterogeneous class, while interacting with the students in a classroom situation, what are the problems which may be faced by a teacher? What should be the immediate action that should be taken by a teacher? He/she may ask various questions of varying difficulty level related to various mathematical concepts, receive responses from each student, record observations about each student and on the basis of the responses, teacher can identify strength and weakness of each student. He may then divide the students into groups accordingly. In each such group, a monitor can be selected on the basis of good academic performance. The monitor can also record the academic performance of each student in his/her group and results may be given to the concerned teacher. On the basis of the teacher's observations and record as well as the group monitors' and classmates' opinion, the students may be identified based on their competencies and performances. These interaction may foster the cooperative learning in the classroom situation. Also there are certain means of recognising academically bright/gifted students and academically challenged students. Some of the problems related to academically challenged students are given below:

(1) There may be some physical ailments, such as poor sightedness, poor hearing, frequently stomach upsetness, headache, etc. which hamper child's studies. The remedy of all such weaknesses can be medically treated.

(2) Some weaknessess as may be due to some mental causes. These causes may be inborn or acquired from environment due to disstisfaction, domestic problem, etc. A case of simple mental problem can be tackled with some chance of success by the teacher. But complicated cases will have to be passed on to a psychologist.

(3) Dislike for the subject may be another weakness. This may be natural or acquired. If it is inborn, teacher's effort may go waste. If it is an acquired one, it may be mostly the teacher's fault. Innovative teaching and positive reinforcement by the teacher can help learner to develop interest in mathematics.

(4) Influence of home may also create distaste and weakness. Some parents unintentionally provide negative suggestions to their children. Some of them are in the habit of saying that they never liked mathematics or they never wanted to study it or were never able to pass in it or that failure in mathematics has been their family tradition. Parents must be made conscious of its adverse effect and their duty in this matter.

(5) The change of school or even change of teacher may not suit some of the students and they become weak in this subject. The situation should be tackled cautiously. Changing the teacher in the middle of a session should be avoided as far as possible.

(6) Students sometimes do not like teacher's routine method of teaching. Students always like newness and novelty. The teacher should always be prepared to adjust his method to the learner's likes.

(7) Some students may need more practice and exposure than others. The teacher should not overlook their needs. Some students may understand a concept using one context problem whereas the others may need more variety and exposure to grasp it. The teacher has to be resourceful to create opportunities to meet learning needs of all students.

9.2.2 How to Identify Strengths and Weaknesses of the Learner

As stated above, for successful teaching in mathematics, the teacher has to identify the strengths and weaknesses of learners. Given below are some methods of identifying childern's knowledge, interests, skills and behaviours. A competent teacher with his experiences and interaction with the child can quickly identify the strengths and weakness of the child for the learning of mathematics.

1. **Observation Method:** In this method, the teacher observes the activities of the child cautiously and carefully. Activities of the child may be reading, writing, doing some work, playing, etc. He/she observes how child accomplishes his/her work. By observation, he/she studies child's interests, understandings, skills and behaviour. The teacher personally attends each child and provides help in solving his/her problems. When the child is playing, the teacher observes whether the child is abiding by the rules of the game or not. When the child is doing some work, the teacher observes in which work the child takes more interest. Gradually, he/she finds the progress of each child.

   Observation can be done in two ways. In the first case, the teacher observes the activities of the child silently. The child does not know that the teacher is watchful of all of his activities. The teacher finds the interest, aptitude, behaviour, etc. and with the passage of time, he/she goes on finding the progress of the child in various spheres.

   In the second case, the teacher asks the child to complete some work under some rules or procedures taught. By his/her work, the teacher concludes whether the child has understood various aspects on completion of the work and whether the child has followed the desired rules and procedures. This type of observation may be called directed observation.

2. **Examination Method:** The strengths and weaknesses of a learner can also be identified by means of asking questions and calling for their answers. These questions may be oral or written or both.

   It should be kept in mind that the aim of the teacher is to find strengths and weaknesses of the child. The teacher should not conduct examination of all the children at one place and time. This can be done in groups or for each child separately. The time of answering also may differ from child to child. Otherwise, some children may develop frustration and fear for mathematics. The teacher must avoid generation of any kind of fear in the mind of the child.

3. **Tests and Interviews:** In this method, the teacher asks the child to conduct some experiment. While doing this, he/she asks relevant/related questions about what has already been taught. Thus, he/she has to ascertain that the child is able to repeat the

knowledge, understanding and skills gained earlier or not. Suppose, the child is asked to find experimentally the sum of all the angles of a quadrilateral. By drawing a diagonal of the quadrilateral, the child can be asked 'What is the sum of all the angles of a triangle?', 'What will be the sum of all the angles of two triangles?'

EXERCISE 9.1

1. What are the main causes of weakness in mathematics? What are the corresponding effective remedies?
2. Weakness in mathematics is not always due to low I.Q. of a student. Comment and discuss.
3. To identify learner's weaknesses by examination method, what precautions should be taken?

9.3 Activities Enriching Mathematics Learning

9.3.1 Assisted Learning

The learning, which is assisted by external means rather than that of the learner himself/herself and apart from the teacher, can be termed as assisted learning. These means may be living as well as non-living things. Not only at higher level, but at lower level too, the learning can be facilitated a lot through external assistance and that too at ease.

Now a days, two types of assisted learning, given below, have proven their importance and are widely used at mass level in professional courses as well as academic courses, at school level as well as at higher level, at formal level as well as at informal level:

(i) Peer assisted learning

(ii) Computer assisted learning.

Peer Assisted Learning

This is a kind of learning, which is assisted by the peers of the learner. In this, the peer group not only observes the learner but also gives feedback and suggestions for improvement, shares some novel ideas and experiences, which in turn facilitate learning.

Computer Assisted Learning

This type of learning is a blend of individualised learning and supported learning. Here, the learner is assisted by the computer. The computer presents content, facilitates learning, presents evaluation tools (for formative as well as summative evaluation), evaluates, gives feedback and strengthening learning. All these activities may be programmed on one end or may be moderated by teacher/expert/mentor. This may be synchronous or asynchronous. A synchronous activity is one, in which both initiator as well as respondent are in contact and interact simultaneously as in chatting, vedio-conferencing, etc. An asynchronous activity is one, in which the initiator and respondent are not interacting simultaneously, e.g. – e-mail, SMS, etc. Thus, one raises query at some time and the other responds at some other time.

Why Assisted Learning?

There are many advantages of assisted learning. Some of these may be as follows:

- Peer assisted learning helps in getting rid of the hesitation of the learner which may occur while interacting with the teacher. Learner feels free to interact with peers than with the teacher/instructor.
- Peer assisted learning provides an extended time duration for learning, as peers may interact in non-formal timings too.
- Peer assisted learning develops a sense of co-operation and co-ordination among peer group.
- While in group, peer may have better and more exhaustive observations to share than that of the teacher alone.
- Peer assisted learning can create an environment of not only formal but informal nature too, which can facilitate in a dual sense for better learning on the part of learner.
- Peer assisted learning develops sense of class community among learners.
- In peer assisted learning, everybody participates and hence, everybody learns simultaneously.
- In computer assisted learning, the learner may learn at his own pace of learning.
- Computer assisted learning provides freedom of time, schedule, media, place and location, interaction, expression, simultaneous multi-tasks, etc. to the learner and as a whole freedom to learn (what to learn, when to learn and how to learn).
- Learner may learn so many things with assisted learning directly, without going through in real life situation. Learner learns not only through his/her own experiences but also through experiences of others.
- Assisted learning paves way for temporal as well as spatial validity to the concepts, views, ideas, experiences, events and hypothetical solutions.
- Assisted learning may ensure learner's active involvement in the learning, his/her acceptance of ideas for improvement and further innovations.

Hence, it can be understood and easily drawn that assisted learning is highly advantageous and fruitful for the learners.

Students are suggested to explore the disadvantages, if any, of assisted learning and how to cope with such disadvantages.

9.3.2 Supplementary Text Materials

One of the important aims of school education is to widen the outlook of the students and to create in them love for extra reading. The students should not be kept confined to the length and breadth of textbook knowledge and classroom teaching. Also, co-curricular activities, group projects and individual work on assignments necessitate the need of materials other than textbooks.

Extra reading makes knowledge complete and comprehensive. It stimulates to choose the best of what has been thought and said. It can help in making a subject lively and interesting. Moreover, a textbook alone cannot satisfy the desire of brilliant students for more and more information.

Supplementary materials are extra materials which supplement a textbook in areas of text prescribed in it. Each of the following may be regarded as a supplementary text material:

1. Exemplar problems based on the textbook
2. Teacher's Handbook (Teacher's Guide) based on the textbook
3. Reference books
4. Enrichment materials (often being part of Teacher's Handbook)
5. Manuals for mathematics laboratory and mathematics kit.

EXERCISE 9.2

1. Mainly which activities need extra materials other than textbooks?
2. What is the role of supplementary text materials in school education?
3. What are the main supplementary text materials?
4. By taking a practical incident from your classroom interaction with your peer group, develop the concept of peer assisted learning. How is it different from computer assisted learning ?

9.3.3 Summer Programmes

The term 'summer programme' is self explanatory. It refers to the educational programmes in which students participate during summer vacations. The period of summer vacations is best suited for students, because it does not disturb regular classroom learning. This helps in optimum utilisation of time resource as most of the schools are closed during summer vacations. Students are free from the burden of formal routine work studies during this time. Therefore, basic idea about these programmes has been evolved to utilise this free time of learners for learning activities in a best possible way.

9.3.4 Need and Importance of Summer Programmes

Students are kept busy with their formal studies throughout the year. They hardly get time for other curricular activities of their own choice and interest. Moreover, every learner has his/her own pace of learning and formal system of schooling creates a sense of hindrance in the overall learning of the learner. In formal system of learning, every learner has to undergo the same preplanned/programmed processes for learning, irrespective of their capabilities and challenges, simultaneously. Consequently, a gap originates in the learning of different learners. There is a need to minimise this gap, as well as, a need to provide opportunities for improving learning by all so as to bring them at equal footing in best possible way. Most of the concerned educationists found summer time as the most appropriate time for this purpose. Hence, a need for summer programmes was felt and the concept was developed. Even today, the summer programmes are equally beneficial for enhancement of knowledge of students.

9.3.5 Objectives of Summer Programmes

In view of the above, main objectives of summer programmes are:

(i) To utilise leisure time of summer vacations of students for learning activities.

(ii) To increase the subject matter competence of students

(iii) To bring students in contact with modern developments

(iv) To motivate and create greater enthusiasm in students for learning of mathematics

(v) To provide opportunities to students for recreational activities in mathematics

(vi) To provide opportunity to students for individualised learning at their own pace

(vii) To provide opportunity to students for enhanced learning in their field of interest

(viii) To enable students for advanced learning in mathematics and develop their special talents

(ix) To raise the standard of learning as a whole and strengthen learners' vigour and creativity.

9.3.6 Organisation of Summer Programmes

Various bodies like Indian Institute of Science, Jawahar Lal Nehru Centre for Advanced Scientific Research and Homi Bhabha Centre for Science Education organise summer programmes for students and professionals of sciences, including mathematics.

While organising a summer programme at a particular place, it is essential that there are facilities of appropriate classrooms, blackboards, library, projector, television, computer, accommodation, etc.

Given below are some suggestions for organisation of successful summer programmes:

1. They should be planned sufficiently in advance by a group of competent mathematics teachers/mathematicians.
2. Summer programmes may be divided into three main categories:
   (i) for specialised learning
   (ii) for improvement in individualised learnings
   (iii) for research/projects.
3. The pre-requisites like detailed syllabi, identification of needs, text/reference books, etc. should be announced in sufficient advance.
4. There should be provision for providing lecture-notes and other related materials to learners.

Though summer programmes are highly beneficial, but they are mainly focussed towards students attached to formal system of education. They seldom take care of those learners who are some how not attached to the formal system of education. This limitation has been tried to cope by correspondence courses (distance education provisions).

9.3.7 Correspondence Courses (Distance Education)

Distance Education courses are run throughout the year, but meant for those who are unable to take up that particular course through regular courses of studies.

There is a large number of learners who wish to learn and want to achieve academic progress, but due to several constraints, such as socio-economic conditions etc., they are unable to attach themselves with the regular courses of study. Correspondence, i.e., distance education courses are remedy for such a problem. Such courses provide opportunity to learners to learn throughout the life. These courses are more flexible and more facilitating than those of regular course of study. Due to the flexibility, such a course gives to the learners freedom not only to move up at their own pace, but also a bit of flexibility in number of chances to appear in terminal examinations.

The novelty of these courses is that they provide not only opportunity to ambitious learners to move up, but also a formal degree for coping up with the degree holders from regular system of education.

9.3.8 Objectives of Distance Education Courses

(i) To provide opportunity to learners who are not able to attach themselves to formal system of education

(ii) To provide opportunity to learners to continue their learning irrespective of their socio-economic conditions

(iii) To provide opportunity to learners to acquire formal degree without attending formal classes

(iv) To provide more flexibility to learners for choosing individual freedom and pace in learning

(v) To provide learners more temporal and spatial freedom for learning

9.3.9 Methodology

Need based distance education lessons, with provision for regular feedback could be taken by the learners who are earnest about upgrading themselves and may lead to certificates, diplomas and degrees. These lessons could, at one end, relate to discussion of special needs of children and at the other end to other advanced courses in the content of mathematics. Well defined distance education lessons/modules could supplement the learnings through formal system.

Directorates/Schools of distance education studies of various organisations and a number of other similar bodies prepare and execute need based distance education-cum-broadcast lessons and also softwares for picking up at T.V. relay stations anywhere in the country, including teleconferencing. Use of computers in the present computer age is of utmost significant. Developed relevant software programmes are proving most useful for enriching learning in mathematics.

9.3.10 Mathematics Clubs

Mathematics club provides an excellent means of stimulating and fostering study of mathematics. It offers excellent opportunity for free discussions of matters of special interest to its members without the necessity of having any particular sequence of topics. The programmes of mathematics clubs cover a wide range of topics. The students get opportunities of dealing with mathematical projects, mathematical games, puzzles, discussions and debates. Mathematics club is, thus, a medium of developing students' interest in mathematics and hence is a useful activity for enriching learning of mathematics.

9.3.11 Need and Importance of Mathematics Club

Mathematics club can serve many purposes as given below:

1. It can be a good medium of developing students' interest in mathematics.
2. It is a suitable forum for organising various mathematical activities.
3. It provides the students an opportunity of free discussions and thus benefitting from experiences of one another.

4. It can help in proper utilisation of leisure time.
5. Club activities help in linking mathematics with practical day-to-day life.
6. It can act as a supplement to classroom teaching.
7. Scholars and expert teachers can be invited through mathematics club for delivering lectures on important topics.
8. Students get opportunities to clarify their doubts in complex situations through mathematics club.
9. Activities of club provide opportunity of leadership, active participation and cooperative working with joint responsibility.
10. It provides opportunity to come in contact with extra-curricular literature of mathematics, such as mathematical magazines, journals, etc.
11. It is a good forum for exchange of mathematical informations, experiences, experiments and innovations.
12. It can organise mathematical competitions, visits of mathematical interests and arrange film shows for students of mathematics.

9.3.12 Organisation of Mathematics Clubs

Mathematics clubs should be run by the students under the supervision and guidance of their subject teacher(s). For proper running of the club, there should be a draft constitution of the club prepared by mathematics teachers in consulation with the Head of the institution. The draft constitution should provide all important details, such as its name, aims and objectives, qualifications of members, membership fee, selection of office bearers, etc.

In the organisational set-up of the club, the Head of the institution may be the patron and senior mathematics teacher as the incharge. All other teachers of the subject should be the staff advisers. Other teachers interested in mathematical activities may be included as associate staff advisers. The membership of the club should be open to all the students of mathematics of the institution.

Mathematics club should have an elected executive body to undertake various activities. The following office bearers may be elected or nominated amongst its members:

(i) President
(ii) Secretary
(iii) Joint Secretary
(iv) Treasurer
(v) Representative of each class or section.

Senior mathematics teacher or the Head of the institution at times may act as Chairperson.

9.3.13 Important Activities of Mathematics Club

1. To prepare mathematical aids and illustrations.
2. To organise recreational activities, such as games, puzzles, riddles, etc. in mathematics.
3. To prepare and collect charts, models, pictures, magazines, journals, etc.
4. To organise discussions, debates, projects, experiments, etc. on relevant topics in mathematics.
5. To organise seminars, career courses and competitions in mathematics.
6. To organise lectures by scholars and teachers of mathematics
7. To organise lectures by experts of allied subjects and professions.
8. To arrange mathematical shows, exhibitions, etc.
9. To arrange certain field trips of mathematical interest like excursions and visits to banks, post offices, business concerns, etc.
10. To organise celebration of important dates and events pertaining to eminent mathematicians and history of mathematics.

9.3.14 Contribution of Activities of Mathematics Club in Effective Teaching-Learning

1. Activities of mathematics club can cover a wide variety of topics related to the subject. If the students participate in such activities whole- heartedly, the club goes a long way in arousing and maintaining the interest of students in the subject. Thus, the teaching and learning of mathematics becomes more meaningful and effective.
2. Various charts, models, graphs and improvised apparatus prepared by members of the club may be used as important teaching aids for teaching-learning of mathematics in the classrooms. This helps in motivating students to learn mathematics.
3. The activities of the club provide an opportunity to the students to express their talents and abilities which creates a healthy competitive atmosphere for learning mathematics.
4. The arrangement of mathematical excursions and visits develops more enthusiasm and interest in the subject.
5. Various activities of the club like debates and discussions on projects and experiments are very much helpful in developing mathematical understanding, skills in deciding wrong and right, power of thinking and reasoning, ability to distinguish between relevant and irrelevant, logical and illogical, etc. This creates love and appreciation for mathematics.

6. Through mathematics club, the students come in contact with renowned persons of the subject. Sometimes, they consider them as their ideals. This helps in making them self-motivated. Thus, more meaningful and effective teaching-learning situation is generated.

9.3.15 Contests and Fairs

Contest can be explained in various senses. This can be taken as, "An occasion on which a winner is selected from among two or more contestants." In a more generalised way, it can be understood as:

"An event in which two or more individuals or teams engage in competition against each other, often for a prize or similar incentive."

In a mathematical contest, two or more individuals compete in a mathematical event. The characteristics of such a programme are:

(i) It is an organised event

(ii) It is a competition

(iii) It is essential for a prize or title.

In mathematics, a very common example of contest is Mathematics Olympiad competition. In this event, students compete at various sequential levels to reach international level. At each level, the winners may receive some prizes or similar incentives.

Another example may be taken as *national talent search examination*, in which mathematics too has been given a due weightage. It is also organised at various levels.

"A fair is a gathering of people to display or trade some produce or other goods and often to enjoy associated funfair entertainment."

In this sense, a fair can be characterised by gathering, display, trade (barter or buy or sell) and enjoyment. In schools, it may often be seen in the form of a fete, where students prepare some goods, either at home or place of the fair; they exhibit these goods and others buy these goods. In a fair or fete, not only students, who sell the goods, but those too who buy the goods, both enjoy and get entertained.

In a mathematical fair or fete, some amazing mathematical figures, charts, games, puzzles, models, etc. can be sold. At the same time, some activities pertaining to mathematics, appreciation, enjoyment, amusement, etc. can be organised for participation of students.

EXERCISE 9.3

1. What do you understand by a summer programme?
2. What is the need of a summer programme for academic development of a learner?
3. Throw light upon objectives of 'summer programme' keeping in mind its applications.

4. What facilities are required for a summer programme in mathematics to be successful?
5. Prepare a report on summer programmes in mathematics held in the last five years.
6. Compare summer programmes and distance education courses.
7. Who are often the members of a mathematics club?
8. State important activities of a mathematics club.
9. What is the contribution of a mathematics club in effective teaching-learning of mathematics?
10. Describe at least two activities, each of which you can plan for contest and fair at your school. Discuss various steps for organising these activities. How do contest and fair help in learning mathematics?

9.3.16 The Laboratory Approach of Teaching Mathematics

The laboratory approach of teaching mathematics provides students with the opportunity to 'discover' mathematics through exploration and visualisation. Activities may be designed which enhance the students' understanding of the subject as taught in the classroom and also provides a glimpse of what is beyond. It enables teachers to explore and integrate new and innovative methods of teaching. The primary objective of any laboratory is to perform experiments and the same is true of a Mathematics Laboratory. An 'experiment' in mathematics may be described as an 'exercise' or a 'project' which

- highlights some known concepts based on a well known mathematical theory
- sheds new light on some aspect of the topic being studied
- leads to some original discovery on the part of the students
- focuses on some interesting applications of mathematics to a real life problem.

In the laboratory, students may be exposed to problems, which are exploratory or investigatory in nature. The emphasis is on the process of mathematics rather than the product. This can be achieved by carefully designing projects, experiments and modelling activities based on the mathematics taught in the curriculum. In this approach, learning is achieved through exploration and purposeful play. The teacher's role is that of a facilitator guiding the students in their 'discovery' of mathematics. It may be appropriate to say that the laboratory approach of teaching mathematics combines the benefits of all theories of learning mathematics including use of technology in mathematics education.

9.3.17 Mathematics Laboratory

A student, who can concentrate well, can learn mathematics well. Therefore, to remove mathematics phobia, it is necessary to motivate the students by arousing and maintaining their interest in mathematics. This will create a positive attitude in them towards mathematics.

For this, it is necessary to make the involved abstractions tangible and concrete by developing the mathematical concepts out of direct personal experiences. It is a fact that most of the elementary and basic concepts on which the superstructure of modern mathematical theories rest, were evolved by the human beings out of direct personal experiences with concrete objects. If we analyse a mathematical problem, its comprehension and ultimate solution hinges around a correct perception of the elements of the physical situation involved in the problem. Therefore, by putting something concrete into the hands of the students, will give a better picture of conceptualisation of the problem. Hence, the traditional approach of teaching mathematics in which abstract concepts are usually presented to the students in an authoritarian way should be discouraged and activity approach which stresses the presentation of concrete experiences should be encouraged. Thus, mathematics has to be learnt by *doing* rather than by *reading*. This doing of mathematics gives rise to the need of a suitable place for performing these activities. A well equipped mathematics laboratory is the suitable place for the same which can instantly motivate the students and create an environment to mathematics learning.

9.3.18 The Purpose of Mathematics Laboratory

A mathematics laboratory can foster mathematical awareness, skill building, positive attitudes and learning by doing experiences in different branches of mathematics, such as algebra, geometry, mensuration, trigonometry, coordinate geometry, statistics and probability, etc. It is the place where students can learn certain concepts using concrete objects and verify many mathematical facts and properties using models, measurements and other activities. It will also provide an opportunity to the students to do certain calculations using tables, calculators, etc. and also to listen or view certain audio-video cassettes, remedial instructions, enrichment materials, etc., of his/her own choice on a computer. Thus, it will act as an individualised learning centre for a student. It provides opportunities for discovering, remedial instruction, reinforcement and enrichment. Mathematics laboratory will also provide an opportunity for the teacher to explain and demonstrate many mathematical concepts, facts and properties using concrete materials, models, charts, etc. The teacher may also encourage students to prepare similar models and charts using materials like thermocol, cardboard, etc., in the laboratory. The laboratory will also act as a forum for the teachers to discuss and deliberate on some important mathematical issues and problems of the day. It may also act as a place for teachers and the students to perform a number of mathematical celebrations and recreational activities. Thus, the purpose of a mathematics laboratory is to enable:

- a student to learn mathematics with the help of concrete objects and to exhibit the relatedness of mathematics with everyday life
- a student to verify or discover some geometric properties using models, measurements, paper cutting, paper folding, etc

- a student to use different tables and ready reckoners in solving some problems
- a student to draw graphs and do certain calculations using computers and calculators
- the students to do some field work like surveying, finding heights, making badminton courts, etc., using instruments kept in the laboratory
- the students to listen or view certain audio or video cassettes and CDs relating to different mathematical concepts/topics
- a student to see a certain programme on a computer as a part of remedial instruction or enrichment under the proper guidance of the teacher
- the students to perform certain experiments
- the students to do certain projects under the proper guidance of the teacher
- the students to perform certain recreational activities in mathematics
- a student to understand visually some abstract concepts by using three-dimensional models
- a student to observe certain concepts and patterns using charts and models
- a student to reinforce the truth of certain algebraic identities using different models
- the students to consult good reference mathematics books, journals, etc., kept in the laboratory
- the teachers to meet and discuss important issues relating to mathematics from time to time
- a teacher explain certain concepts, data, graphs, etc., using slides
- a teacher to generate different sets of parallel tests using a computer for testing the achievement of students
- the budding mathematicians to take inspiration from the lives, works and anecdotes relating to great mathematicians.

9.3.19 Role of Mathematics Laboratory in Teaching-Learning

Mathematics is a compulsory subject at the Secondary Stage. Access to quality mathematics education is the right of every child. Mathematics engages children to use abstractions to establish precise relationships, to see structures, to reason out things, to find truth or falsity of statements (NCF–2005). Therefore, mathematics teaching in schools must be planned in such a way that it should nurture the ability to explore and seek solutions to problems of not only the academic areas but also of daily life. In order to do this, access to laboratory is an essential requirement which our system has not been able to provide so far. It is proposed to fill this gap by providing a mathematics laboratory at the secondary stage to all the schools. This facility will bring about a renewed thrust in our schools so far teaching-learning of mathematics is concerned.

The teaching-learning of mathematics needs to be characterised by focussed emphasis on processes, such as activity-based learning, making observations, collection of data, classification, analysis, making hypothesis, drawing inferences and arriving at a conclusion for establishing the objective truth.

The higher aim of mathematics education is to 'develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstractions' (NCF–2005). To achieve this, a variety of methods and skills have to be adopted in the teaching-learning situations. The basic arithmetical skills offered in the first eight years of schooling will stand in good stead to achieve the above aim visualised at the secondary stage. A stronger emphasis is to be laid on problem-solving and acquisition of analytical skills in order to prepare children to tackle a wide variety of life situations. Abstraction, quantification, analogy, case analysis, guesses and verification, exercises are useful in many problem-solving situations (NCF–2005). Another area of concern which teachers will have to address is of the perceived 'stand-alone' status that mathematics vis-a-vis other subject areas has in the school curriculum.

One of the biggest challenges of a mathematics teacher is to create and sustain interest in his students. There is a general feeling that mathematics is all about formulas and mechanical procedures. Under these circumstances, a mathematics laboratory will help teachers to reorient their strategies and make mathematics also an activity-oriented programme in schools.

In the huddled and bundled classroom situations, it is indeed difficult to make complex theoretical concepts very clear to all the students. Developing the habit of critical thinking and logical reasoning, which is most important in mathematics learning, also suffers under such claustrophobic classroom situations. A mathematics corner in the lower classes and a mathematics laboratory with appropriate tools at the secondary stage will enable children to translate abstractions into specific figures, shapes and patterns that will provide opportunities to visualise abstractions with greater ease. To promote interest in the subject, mathematics laboratory has become a reality at many places and is considered as an established strategy for mathematics teaching-learning. Since a practical exercise takes a longer time than a theoretical solution might require, it gives the student additional time for better assimilation leading to stronger retention. For students in whom aptitude for mathematics is limited, practical activities may help create positive attitude and a new thirst for knowledge.

9.3.20 Management and Maintenance of Laboratory

There is no second opinion that for effective teaching and learning 'Learning by Doing' is of great importance as the experiences gained remains permanently affixed in the mind of the child. Exploring what mathematics is about and arriving at truth provides for pleasure of doing, understanding, developing positive attitude and learning processes of mathematics and above all the great feeling of attachment with the teacher as facilitator. It is said, 'a teacher teaches the truth but a good teacher teaches how to arrive at the truth'.

A principle or a concept learnt as a conclusion through activities under the guidance of the teacher stands above all other methods of learning and the theory built upon it, is usually not forgotten. On the contrary, a concept stated in the classroom and verified later on in the laboratory doesn't provide for any great experience nor make child's curiosity to know any good nor provides for any sense of achievement.

A laboratory is equipped with instruments, apparatus, equipments and models apart from facilities like water, electricity, etc. Non availability of a single material or facility out of these may hinder the performance of any experiment/activity in the laboratory. Therefore, the laboratory must be well managed and well maintained.

Management and maintenance of a laboratory may be classified into two categories namely the personal management and maintenance and the material management and maintenance.

(A) Personal Management and Maintenance

The persons who manage and maintain laboratories are generally called laboratory assistants and laboratory attendants. Collectively they are known as laboratory staff. Teaching staff also helps in managing and maintenance of the laboratory whenever and wherever it is required.

In personal management and maintenance, the following points are worth consideration:

1. **Cleanliness:** A laboratory should always be neat and clean. When students perform experiments/activities during the day, it certainly becomes dirty and things are scattered. So, it is the duty of the lab staff to clean the laboratory regularly when the day's work is over and also place the things at their proper places if these are lying scattered.
2. **Checking and Arranging Materials for the Day's Work:** Laboratory staff should know what activities are going to be performed on a particular day. The material required for the day's activities must be arranged one day before.

   The materials and instruments should be arranged on tables before the class comes to perform an activity or the teacher brings the class for a demonstration.

3. The facilities like water, electricity, etc. must be checked and made available at the time of experiments.
4. It is better if a list of materials and equipments is pasted on the wall of the laboratory.
5. Many safety measures are required while working in laboratory. A list of such measures may be pasted on a wall of the laboratory.
6. While selecting the laboratory staff, the school authority must see that the persons should have their education with mathematics background.
7. A training of 7 to 10 days may be arranged for the newly selected laboratory staff with the help of mathematics teachers of the school or some resource persons outside the school.
8. A first aid kit may be kept in the laboratory.

(B) Management and Maintenance of Materials

A laboratory requires a variety of materials to run it properly. The quantity of materials, however, depends upon the number of students in the school.

To manage and maintain materials for a laboratory following points must be considered:

1. A list of instruments, apparatus, activities and material may be prepared according to the experiments included in the syllabus of mathematics.
2. A group of mathematics teachers may visit the agencies or shops to check the quality of the materials and compare the rates. This will help to acquire the materials of good quality at appropriate rates.
3. The materials required for the laboratory must be checked from time to time. If some materials or other consumable things are exhausted, orders may be placed for the same.
4. The instruments, equipments and apparatus should also be checked regularly by the laboratory staff. If any repair is required it should be done immediately. If any part is to be replaced, it should be ordered and replaced.
5. All the instruments, equipments, apparatus, etc. must be stored in the almirahs and cupboards in the laboratory or in a separate store room.

9.3.21 Some Activities in Mathematics

Activity 1 : To verify the algebraic identity

$$(a + b)^2 = a^2 + 2ab + b^2$$

Material Required : Drawing sheet, cardboard, cello-tape, coloured papers, cutter and ruler.

Method of Construction

1. Cut out a square of side length *a* units from a drawing sheet/cardboard and name it as square ABCD [see Fig. 9.1].
2. Cut out another square of length *b* units from a drawing sheet/cardboard and name it as square CHGF [see Fig. 9.2].
3. Cut out a rectangle of length *a* units and breadth *b* units from a drawing sheet/ cardbaord and name it as a rectangle DCFE [see Fig. 9.3].
4. Cut out another rectangle of length *b* units and breadth *a* units from a drawing sheet/ cardboard and name it as a rectangle BIHC [see Fig. 9.4].

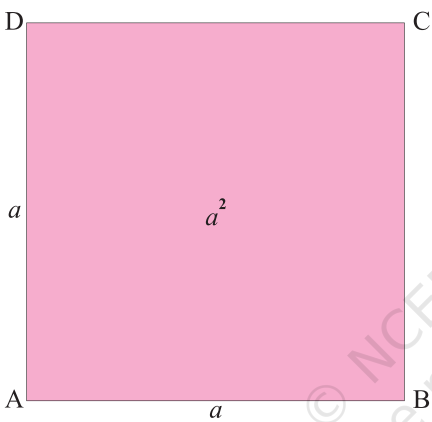


Fig. 9.1

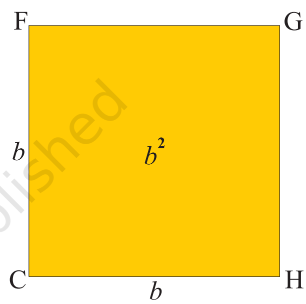


Fig. 9.2

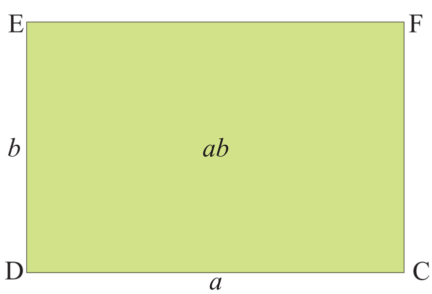


Fig. 9.3

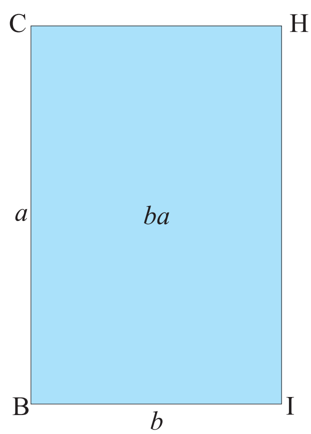


Fig. 9.4

5. Total area of these four cut-out figures

   = Area of square ABCD + Area of square CHGF + Area of rectangle DCFE + Area of rectangle BIHC

   $= a^2 + b^2 + ab + ba = a^2 + b^2 + 2ab$.

6. Join the four quadrilaterals using cello-tape as shown in Fig. 9.5.

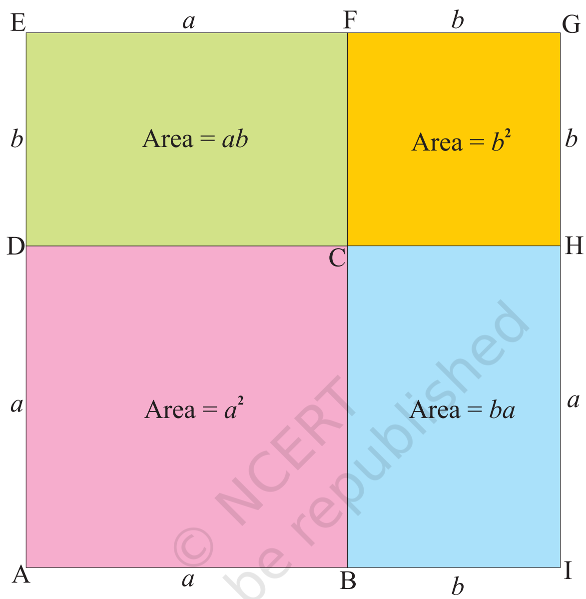


Fig. 9.5

Clearly, AIGE is a square of side $(a + b)$. Therefore, its area is $(a + b)^2$. The combined area of the constituent units $= a^2 + b^2 + ab + ab = a^2 + b^2 + 2ab$.

Hence, the algebraic identity $(a + b)^2 = a^2 + 2ab + b^2$

Here, area is in square units.

Observation

On actual measurement:

a = ..., b = ...$(a+b)$ = ...,

So, a^2 = ... b^2= ...., ab = ...

$(a+b)^2$ = ..., $2ab$ = ...

Note: The above exercise should be repeated twice by taking two different measurements of *a* and *b*.

Therefore, $(a+b)^2 = a^2 + 2ab + b^2$.

The identity may be verified by taking different values of *a* and *b*.

Activity 2 : To verify that the sum of angles of a triangle is 180°

Material Required

Drawing sheets, cardboard, coloured papers, scissors, ruler and adhesive.

Method of Construction

1. Take a hardboard sheet of a convenient size and paste a white paper on it.
2. Cut out a triangle from a drawing sheet, and paste it on the hardboard and name it as ΔABC .
3. Mark its three angles as shown in Fig. 9.6
4. Cut out the angles respectively equal to ∠A, ∠B and ∠C from a drawing sheet using tracing paper [see Fig. 9.7]

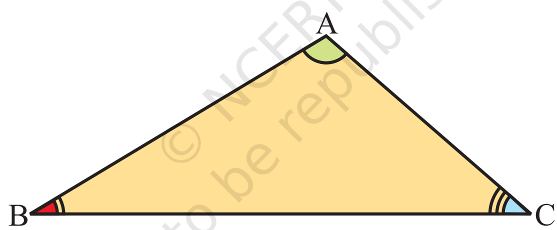


Fig. 9.6

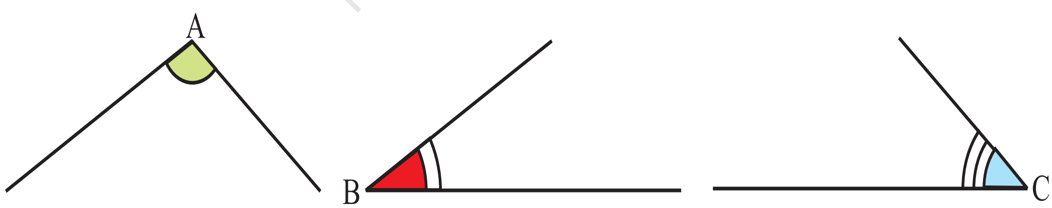


Fig. 9.7

5. Draw a line on the hardboard and arrange the cut-outs of three angles at a point O as shown in Fig. 9.8.

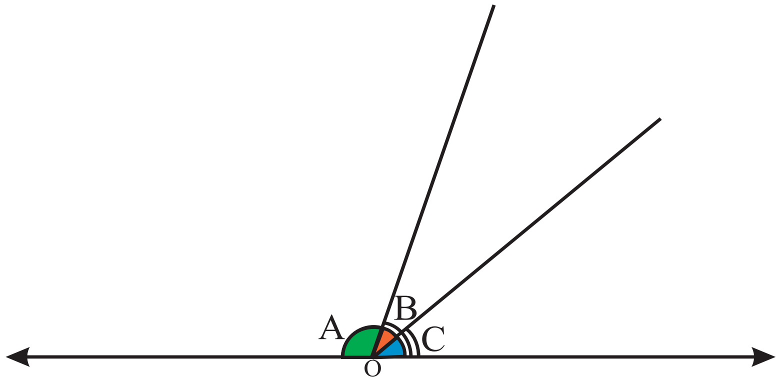


Fig. 9.8

Demonstration

The three cut-outs of the three angles A, B and C placed adjacent to each other at a point form a line forming a straight angle = 180°.

Note: The above exercise may be repeated by taking another different triangle.
It shows that sum of the three angles of a triangle is 180º, i.e., $\angle A + \angle B + \angle C = 180°$.

Observation

Measure of $\angle A$ = _______________.

Measure of $\angle B$ = _______________.

Measure of $\angle C$ = _______________.

Sum ($\angle A + \angle B + \angle C$) = __________.

Activity 3 : To find the formula for the area of a trapezium experimentally

Material Required

Hardboard, thermocol, coloured glazed papers, adhesive and scissors.

Method of Construction

1. Take a piece of hardboard for the base of the model.
2. Cut two congruent trapeziums of parallel sides *a* and *b* units [see Fig. 9.9].
3. Place them on the hardboard as shown in Fig. 9.10.

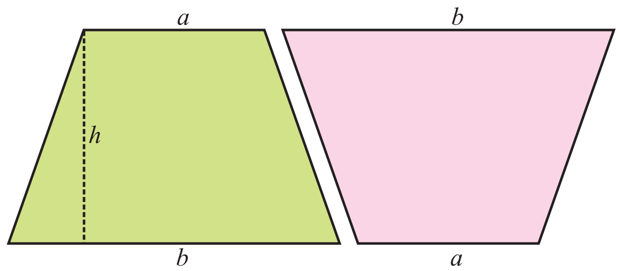


Fig. 9.9

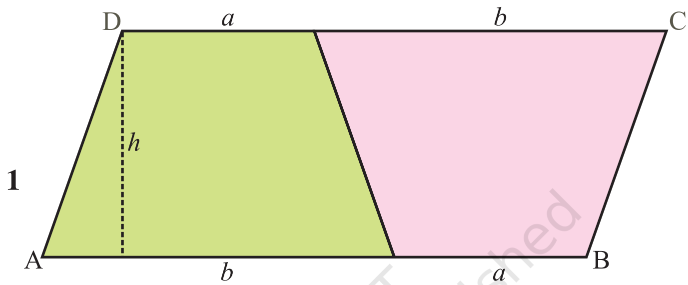


Fig. 9.10

Demonstration

1. Figure formed by the two trapeziums [see Fig. 9.10] is a parallelogram ABCD.
2. Side AB of the parallelogram = $(a + b)$ units and its corresponding altitude = h units.
3. Area of each trapezium $= \frac{1}{2}$ (area of parallelogram) $= \frac{1}{2}(a+b)\times h$

   Therefore, area of trapezium $= \frac{1}{2}(a+b)\times h$

   $= \frac{1}{2}$ (sum of parallel sides) × perpendicular distance.

Here, area is in square units.

Observation

Lengths of parallel sides of the trapezium = --------, --------.

Length of altitude of the parallelogram = --------.

Area of parallelogram = ---------------.

Area of the trapezium $= \frac{1}{2}$ (Sum of ---------- sides) × ---------.

Activity 4 : To find a formula for the curved surface area of a right circular cylinder, experimentally.

Material Required

Coloured chart paper, cello-tape, ruler.

Method of Construction

1. Take a rectangular chart paper of length l units and breadth b units [see Fig. 9.11].

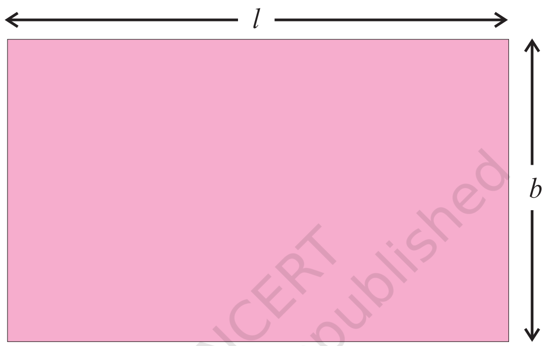


Fig. 9.11

2. Fold this paper along its breadth and join the two ends by using cello-tape and obtain a cylinder as shown in Fig. 9.12.

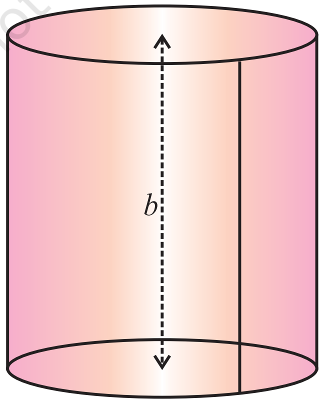


Fig. 9.12

Demonstration

1. Length of the rectangular paper = l = circumference of the base of the cylinder = $2\pi r$, where r is the radius of the cylinder.
2. Breadth of the rectangular paper = b = height (h) of the cylinder.
3. The curved surface area of the cylinder is equal to the area of the rectangle = $l \times b$ = $2\pi r \times h = 2\pi rh$ square units.

Observation

On actual measurement :

l = ..., b = ...,

$2\pi r = l$ = ..., $h = b$ = ...,

Area of the rectangular paper = $l \times b$ = ...

Therefore, curved surface area of the cylinder = $2\pi rh$.

Activity 5 : To find the sum of first *n* natural numbers

Material Required

Cardboard, coloured papers, white paper, cutter, adhesive.

Method of Construction

1. Take a rectangular cardboard of a convenient size and paste a coloured paper on it. Draw a rectangle ABCD of length 11 units and breadth 10 units.
2. Divide this rectangle into unit squares as shown in Fig. 9.13.
3. Starting from upper left-most corner, colour one square, 2 squares and so on as shown in the figure.

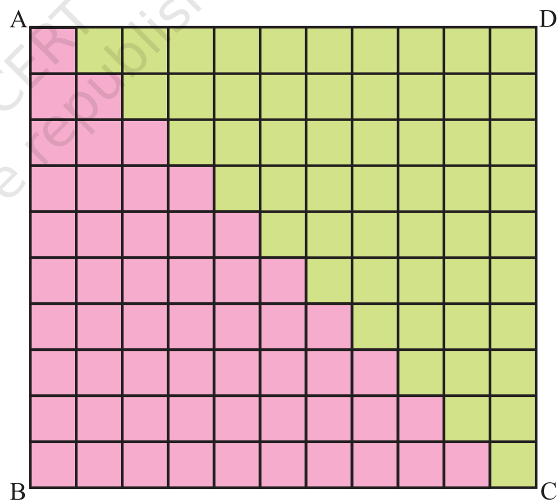


Fig. 9.13

Demonstration

1. The pink colour region looks like a stair case. Also, coloured region is equal to uncoloured region.
2. Length of 1st stair is 1 unit, length of 2nd stair is 2 units, length of 3rd stair is 3 units, and so on, length of 10th stair is 10 units.

3. These lengths give a pattern

   1, 2, 3, 4, ..., 10,

   which is an AP with first term 1 and common difference 1.

4. Sum of first ten terms

   $= 1 + 2 + 3 + ... +10 = 55$ (1)

   Area of the shaded region $= \frac{1}{2}$ (area of rectangle ABCD)

   $= \frac{1}{2} \times 10 \times 11$, which is same as obtained in (1) above. This shows that the sum of the first 10 natural numbers is $\frac{1}{2} \times 10 \times 11 = \frac{1}{2} \times 10(10+1)$.

   This can be generalised to find the sum of first n natural numbers as

$$S_n = \frac{1}{2} n(n+1) \quad (2)$$

Observation

For $n = 4$, S_n = ...

For $n = 12$, S_n = ...

For $n = 50$, S_n = ...

For $n = 100$, S_n = ...

EXERCISE 9.4

1. Using concrete objects, verify the following algebraic identities:

   (i) $(a - b)^2 = a^2 - 2ab + b^2$ (ii) $a^2 - b^2 = (a + b)(a - b)$

2. Verify that an exterior angle of a triangle is equal to the sum of the two interior opposite angles of the triangle by an activity.

3. Form a cone from a sector of a circle and find the formula for its curved surface area by conducting an activity.

4. Find the sum of first n odd natural numbers with the help of an activity.

5. Verify Pythagoras theorem by performing an activity.

9.3.22 Recreational Activities

There are a number of myths prevalent about mathematics. Some of them may have some modicum of truth in them but there is a terrible exaggeration in most of them. One myth is that mathematics is a cold, dry and uninteresting subject. This misconception occurs because of the failure on the part of teachers and textbooks to stimulate and maintain interest in mathematics. A teacher can strive to make mathematics interesting by tactful presentation of the subject matter. This can be done mainly

(i) by focussing on the development of real understanding

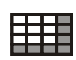
(ii) by introducing the elements of recreation in the forms of patterns, games, magic squares, riddles, puzzles, etc. and

(iii) by including applications of mathematics in other school subjects.

Here, we shall focus our attention on (ii).

Mathematics is full of recreations. It depends on the teacher how he/she makes their use in enriching learning mathematics. Number patterns are interesting to observe and one may be curious enough to find their rationales. Apart from being recreational, they encourage an alert and open minded attitude in youngsters and help them in developing high degree of logical thinking. Consider for example, the following patterns:

(1) **The sum of first n odd numbers**

					Rationale	
1	(one term)	=	1^2	$\rightarrow$		$\leftarrow$ 1
1+3	(two terms)	=	2^2	$\rightarrow$		$\leftarrow$ 1+3
1+3+5	(three terms)	=	3^2	$\rightarrow$		$\leftarrow$ 1+3+5
1+3+5+7	(four terms)	=	4^2	$\rightarrow$		$\leftarrow$ 1+3+5+7
$\vdots$	$\vdots$		$\vdots$			
1+3+5+7+...	(n terms)	=	n^2		and so on.	

(2)
$0 \times 9 + 1 = 1$
$01 \times 9 + 2 = 11$
$12 \times 9 + 3 = 111$
$123 \times 9 + 4 = 1111$
$1234 \times 9 + 5 = 11111$
and so on.

Rationale : Consider $1234 \times 9 + 5$

$$
\begin{aligned}
1234 \times 9 + 5 &= (1111 + 111 + 11 + 1) \times 9 + 5 \\
&= (9999 + 999 + 99 + 9) + 5 \\
&= (9999 + 1) + (999 + 1) + (99 + 1) + (9 + 1) + 1 \\
&= 10000 + 1000 + 100 + 10 + 1 \\
&= 11111
\end{aligned}
$$

Similarly, magic squares create interest. In ancient China and India, some people used to wear stone or metal ornaments engraved with arrays of numbers. If you add the numbers of any row, column or diagonal, you will get the same sum. Such arrangement of numbers in the form of a square is called a *magic square.*

Since their discovery, magic squares have been the source of many mathematical amusements and games. Also, a number of interesting and useful mathematical concepts have been discovered as the result of research on the theory of magic square containing 3 rows and 3 columns is called a magic square of *order 3*. The common sum obtained by adding the elements of a row, column or diagonal is called the *constant* of the magic square.

Usually magic squares are formed from the consecutive numbers.

8	1	6
3	5	7
4	9	2

Magic square of order 3
Constant of magic square : 15

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Magic square of order 4
Constant of magic square : 34

There are separate ways of constructing magic squares of odd order, even (multiple of 4) order and even (not a multiple of 4) order.

Even shorter ways of calculations create interest in mathematics. When man first invented numbers and methods of counting and computing, he was fascinated with his new ideas. In ancient times, computing was considered such a special art that only a selected number of people could work with numbers. As time passed, the fascination of calculation began to fade and computing began to seem more like work than play. As man acquired more knowledge and as his life became more complex he started to search for ways to shorten the work of computation. This led to the development of computing devices, such as calculating machines and electronic computers of today.

Although many computing devices, have been invented but they cannot completely eliminate the need for good old computing skills. After all, you cannot carry a calculating device with you to perform computations necessary for checking a grocery bill, a bank statement, etc. However, there are many things that can be done to aid in mental computation and cut down on the amount of paper work needed for a calculation. We give here some of such shorter ways of calculations:

Activity 1: Squaring 2 -digit numbers ending in 5

Method : Multiply ten's digit by the next whole number and insert 25 on the right of the product:

$$25^2 = 25 \times 25 = (2 \times 3), 25 = 625;$$

$$65^2 = 65 \times 65 = (6 \times 7), 25 = 4225;$$

Rationale : A 2-digit number ending in 5 can be expressed as $(10n + 5)$ where n is the tens digit of the number.

$$(10n + 5)^2 = 100n^2 + 100n + 25$$

$$= 100n\ (n + 1) + 25$$

↙ shows zeros in one's and ten's places ↘ ten's digit ↘ Next whole number

If we have to multiply two numbers which have equal ten's digits and the sum of whose one's digits is 10, we can use the above method to get their product.

Thus, $24 \times 26 = (2 \times 3), (4 \times 6) = 624;$

$73 \times 77 = (7 \times 8), (3 \times 7) = 5621;$

$56 \times 54 = (5 \times 6), (6 \times 4) = 3024;$ etc.

Activity 2: Use of Distributive Laws

$$97 \times 288 + 3 \times 288 = 288 \times (97 + 3) = 288 \times 100 = 28800;$$

$$489 \times 117 - 17 \times 489 = 489 \times (117 - 17) = 489 \times 100 = 48900;$$ etc.

9.3.23 Games

We usually play games to pass our leisure time. Many number games have been developed in mathematics, which apart from killing the leisure time create interest and develop positive attitude towards the subjects of mathematics. To clearly understand or play a number game to win, one has to go to the minute details in the process of the game and thus this indirectly helps in developing the power of logical thinking and reasoning. To understand why the aspect of the game is important from our point of view.

Game 1 : Who will first make 100?

Two players play this game in which each player speaks a whole number from 1 to 10, both inclusive. They go on speaking by turns and at each stage, the grand total is done. The one, who first makes 100, is declared winner.

Example : Suppose Kamal and Shobha are the players

Speaker	Speaks	Grand Total
Kamal	5	5
Shobha	8	13
Kamal	9	22
Shobha	4	26

and so on till grand total is 100.

If you know the technique and your rival is unaware, you will always be the winner.

Technique : The one who will first make 100, is who first makes 89 (100 – 11). Why?
The one who will first make 89, is who first makes 78 (89 – 11)
and so on, who first makes 67, 56, 45, 34, 23, 12, 1.

Game 2: Game on addition of single digit numbers

Apparatus: Use two dice each numbered from 1 to 6. Each of the two players has a set of 15 counters. One player uses left hand board and the other right hand board which are given below:

1	3	6
9	2	8
5	2	4
3	7	1
6	5	7

1	3	6
9	2	8
5	2	4
3	7	1
6	5	7

Steps of the Game

1. Players take turns to throw the two dice simultaneously.
2. Find the sum of your two dice numbers; suppose it is 7, you can put a counter on number 7 of your board or you can put counters on board numbers whose sum is 7, e.g., 3,4; 1,6; 1,2,4; 2,5; etc.
3. If you cannot put a counter, do nothing.
4. The winner is the first player to cover his/her board.

Game 3 : Tricky way to calculate quickly

Step 1 : Ask your friend to write a 5-digit number : 56249

Step 2 : Ask him to write another 5-digit number : 82467

Step 3 : You write number (orally calculate 99999 – 56249) : 43750

Step 4 : Ask your friend to write one more 5-digit number : 69403

Step 5 : You write number (orally calculate 99999 – 82467) : 17532

Step 6 : Ask him to write another 5-digit number : 32147

Step 7 : You write number (orally calculate 99999 – 69403) : 30596

Ask him, now, to find the sum of these 7 numbers. If he does not know the trick, he will take usual (longer) time in finding the sum.

To get the sum quickly, you take the remaining number 32147, subtract 3 from this number and insert 3 at the left of the result. Thus, the required sum is 332144.

Your friend will be astonished, how you could calculate so quickly.

Rationale : You made 43750 + 56249 = 99999; 82467 + 17532 = 99999; and 69403 + 30596 = 99999. Thus, the required sum is:

$$3 \times 99999 + 32147 = 3 \times (100000 - 1) + 32147 = 300000 + (32147 - 3)$$

Another kind of game is in which you ask your friend to perform certain operations and on the basis of the last disclosed result, you make a forecast of things unknown to you. The game, given below, will clarify.

Game 4 : Guessing the Date and Month of Your Birth

Step 1 : Ask your friend to multiply the date of his/her birth by 20.

Step 2 : Ask to add 73 to the resulting number obtained in Step 1.

Step 3 : Ask him/her to multiply the number obtained in Step 2 by 5.

Step 4 : Ask to add serial number of the month of his/her birth to the number obtained in Step 3.

Let this number be disclosed.

To forecast the date and the month of birth, subtract 365 from the disclosed number. The first digit (in case of 3-digit number) or first two digits (in case of a 4-digit number) on the left of the resulting number make the date of birth and the other two digits make the serial number of the month of the birth.

Example : Suppose the date of birth is 17th August.

Step 1 : $17 \times 20 = 340$

Step 2 : $340 + 73 = 413$

Step 3 : $413 \times 5 = 2065$

Step 4 : 2065 + 08 = 2073 {since serial number of August is 8}

To forecast, we get 2073 – 365 = 1708

This gives date 17 and month 08, i.e., August.

Rationale : Let the date of birth be x and serial number of month be y.

By the above operations, we get $(20x + 73) \times 5 + y = 100x + y + 365$

To forecast, we have $(100x + y + 365) – 365 = 100x + y$

This shows that x represents number of hundreds and y makes ten's and one's places.

9.3.24 Riddles in Mathematics

Questions with clever or surprising answers are popularly called *riddles.* A true riddle always asks a question that can be answered reasonably. Riddles are world's oldest guessing games. Riddles not only create interest in learning of mathematics, they help in the development of logical thinking and reasoning. A sample of few riddles is given below:

Riddle 1 : Nine dots are arranged in a square formation in three rows of 3. Draw 4 straight line-segments, the second beginning where the first ends, the third beginning where the second ends and the fourth beginning where the third ends so that each dot is on at least one line segment.

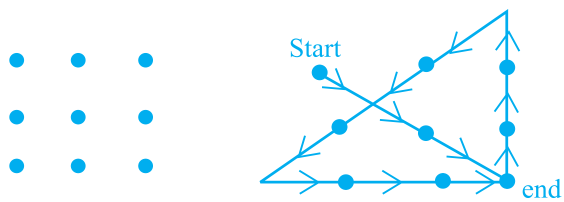
Riddle 2 : A wall clock takes 6 seconds in striking at 6 o'clock. How many seconds does the same clock take in striking at 11 o'clock?

Riddle 3 : A well is 10 metres deep. A frog climbs up 5 metres during the day but falls back 4 metres during the night. Assuming that the frog starts from the bottom of the well, on which day does he get to the top?

Riddle 4 : A can of 8 litres is full of milk. The milkman has only two empty containers which can hold 5 litres and 3 litres of milk respectively. A customer wants 4 litres of milk. How will the milkman measure the milk?

Solutions to Riddles

Riddle 1 :



Riddle 2 : 12 seconds, since for 5 intervals it takes 6 seconds, for 10 intervals it will take 12 seconds.

Riddle 3 : 6th day

Riddle 4 : Fill 3-L container, pour in to 5-L container. Again fill 3-L container and pour into 5-L container. 1-L milk will be left in 3-L container. Empty 5-L container by pouring its milk back in 8-L container. Now, pour 1-L milk of 3-L container into 5-L container. Fill 3-L container again from 8-L container milk and pour it in 5-L container.

Thus, 5-L container will contain 4 litres of milk.

9.3.25 Puzzles

By a puzzle in mathematics, we mean a statement apparently leading to a particular answer/ result often seeming to be impossible whereas by means of deeper reasoning and analysis is found to give a different answer/result altogether. Puzzles are also called brain-teasers. Solutions of puzzles enrich learning of mathematics. They develop power of thinking and reasoning in students. A sample of some puzzles is given below:

Puzzle 1 : Determine a number such that when its right most digit is inserted on its left, it becomes double. (There are more than one solutions.)

Puzzle 2 : The numbers 1, 2, 3, 4, ..., 999, 1000 are multiplied together. How many zeroes will there be at the end (on the right side) of the product?

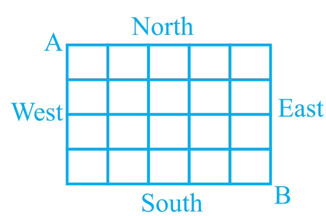
Puzzle 3 : Two railway stations are 100 kilometres apart. Two trains start from these two stations towards each other at the same time and each moves at a uniform speed of 50 km/h.

Just as the trains start, a bird leaves the first train and flies directly to the other train. As soon as the bird reaches the second train, it flies back directly to the first train. As soon as it reaches the first train, it flies back directly to the second train. This goes on until the two trains meet.

If the bird has flown at a uniform speed of 100 km/h, how many kilometres would the bird have flown upto the time of meeting of the trains?

Puzzle 4 : A game is played by three players in which the one, who loses, must double the amount of money that each of the other players has at that time. Each of the players loses one game and at the conclusion of the three games each has Rs 16. With how much money did each man start?

Puzzle 5 : In the diagram, given here, how many routes are there from A to B, if you are not allowed to go north or west?



Solutions to Puzzles

Puzzle 1 : 105263157894736842; 210526315789473684; 421052631578947368

Puzzle 2 : Since 2×5 generates one zero and 5 occurs 249 $\left(\frac{1000}{5}+\frac{1000}{25}+\frac{1000}{125}+\frac{1000}{625}\right.$ *i.e.,* $\left.200+40+8+1\right)$ times as a factor in the product $1\times2\times3\times4\times...\times 999\times1000$, the required number of zeros is 249.

Puzzle 3 : Since the bird flies for 1 hour, it covers, in all, 100 km.

Puzzle 4 : Rs 26, Rs 14 and Rs 8

Puzzle 5 : $126\left(\frac{|\underline{(5+4)}}{|\underline{5}\,|\underline{4}}\right)$ routes since each route contains 5 horizontals and 4 verticals.

EXERCISE 9.5

1. Express each of the numbers from 1 to 30 by expressions having exactly four 4's and any operational symbols (+, –, ×, ÷, $\sqrt{\ }$, etc.)

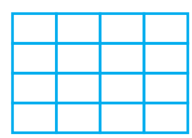
   First three have been done for you:

$$1=\frac{4+4}{4+4} \text{ or } \frac{4\times4}{4\times4} \text{ or } \frac{4\div4}{4\div4} \text{ or } \frac{44}{44}$$

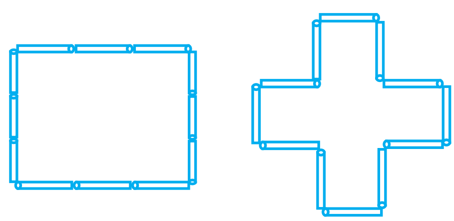
$$2=\frac{4\times4}{4+4} \text{ or } \frac{4\times4}{\sqrt{4}\times\sqrt{4}}$$

$$3=\frac{4+4+4}{4} \text{ or } \frac{\sqrt{4}+\sqrt{4}+\sqrt{4}}{\sqrt{4}} \text{ or } 4-\left(\frac{4}{4}\right)^4$$

2. Fill up the cells of the square, drawn here, by a, b, c, d so that each row, each column and each diagonal contains all of them.



3. Assuming that a match-stick is a unit (cm) of length, it is possible to place 12 match-sticks on a plane surface in various ways to form figures with integral areas. Given here are two figures a square of area 9 cm^2 and a cross of area 5 cm^2, each using all 12 match sticks.



Use all 12 match sticks to form, in similar fashion, a figure having an area of exactly

(i) 6 cm^2 (ii) 4 cm^2

4. A night-watchman was offered initial salary of Rs 10000 or half yearly salary of Rs 5000. He was given an option of a yearly increase of Rs 2000 in first case and half yearly increase of Rs 500 in second case. Which option is more beneficial for him?
5. Three shop-keepers sold 23 pencils, 37 pencils and 30 pencils, respectively. Each sold pencil at the same rate and received the same amount of money from the sale. At what rate were the pencils sold by them?
6. A man has 20 metres of cloth. He wants to prepare exactly 20 clothes from it in the form of shirts, trousers and caps. A shirt requires 3 metres, a trouser 2 metres and a cap half metre of cloth. How many shirts, how many trousers and how many caps will he have to prepare?

ANSWERS

1. $4 = 4 + 4 - \sqrt{4} - \sqrt{4}$; $5 = 4 + \left(\frac{4}{4}\right)^4$; $6 = \frac{4+4+4}{\sqrt{4}}$;

$7 = 4 + 4 - \left(\frac{4}{4}\right)$; $8 = 4 \times 4 - 4 - 4$ $9 = 4 + 4 + 4 \div 4$;

$10 = 4 + \sqrt{4} + \sqrt{4} + \sqrt{4}$; $11 = \frac{44}{\sqrt{4 \times 4}}$; $12 = \frac{44+4}{4}$;

$13 = \frac{44}{4} + \sqrt{4}$; $14 = 4 + 4 + 4 + \sqrt{4}$; $15 = 4 \times 4 - 4 \div 4$;

$16 = 4 + 4 + 4 + 4$; $17 = 4 \times 4 + \frac{4}{4}$; $18 = 4 \times 4 + 4 - \sqrt{4}$;

$19 = \underline{|4} - 4 - \frac{4}{4}$; $20 = 4 \times 4 + \sqrt{4 \times 4}$; $21 = \underline{|4} - 4 + \frac{4}{4}$;

$22 = 4 \times 4 + 4 + \sqrt{4}$; $23 = \underline{|4} - \left(\frac{4}{4}\right)^4$ $24 = 4 \times 4 + 4 + 4$;

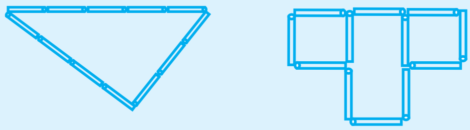
$25 = \underline{|4} + \left(\frac{4}{4}\right)^4$; $26 = \underline{|4} + \frac{4+4}{4}$; $27 = \underline{|4} + 4 - 4 \div 4$;

$28 = \lfloor\underline{4} + \frac{4 \times 4}{4}$; $29 = \lfloor\underline{4} + 4 + \frac{4}{4}$; $30 = \lfloor\underline{4} + \sqrt{4} + \sqrt{4} + \sqrt{4}$

2.

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

3.



4. Second; calculate yearwise **5.** Rs 5 per dozen and Re 1 for a pencil

6. 14 caps, 5 trousers and 1 shirt.

9.3.26 Cooperative Learning

During observation of practice teaching in many schools over many years, it has been invariably found that mathematics teachers tightly holding all the instructional strings with hardly any space provided for learner - initiated dialogue and discussion. This robs young learners of mathematics from playing an active role and dispelling their doubts and bridging gaps in their learning. To provide an opportunity to a learner to play an active role and to develop firm understandings in mathematics, cooperative learning is one of the devices which is discussed below:

9.3.26.1 Cooperative Learning Approach

Cooperative learning approach (CLA) promotes learning in an environment where students in small groups share ideas and work collaboratively to complete academic tasks. The use of CLA in classroom is justified from cognitive perspective because it promotes group discussion which in turn promotes better information processing by learners thereby enabling them to develop clear conceptual understanding. CLA can also be justified from social cognitive perspective because it promotes cooperation-oriented interaction.

CLA is used by teachers in a variety of ways because of the differences in their understanding about the nature of learning and the roles of the students and the teacher in the classroom. In spite of these surface level differences, CLA essentially involves students working in small groups or teams to help one another learn academic materials. Contemporary literature mentions four major models of CLA. These are – Student Team Learning, Jigsaw, Learning Together and Group Investigation.

The essence of these models is as follows:

Student Team Learning (STL) assumes that students work together to learn and are responsible for one another's learning as well as their own. In addition, it emphasises the use of team goals and team success. Consequently, under STL, the students are not just to 'do something as a team', but 'learn something as a team'. The three central concepts in STL are (a) team rewards, (b) individual accountability and (c) equal opportunities of success.

Jigsaw: Here students are divided into 4-5 member teams who work on academic tasks/materials that can be further broken down into as many groups as the number of members in a team. These teams are called 'home teams'. After all members of home teams have got their specific assignments, they break up and form 'expert teams'. Here members discuss and analyse their tasks, collect additional evidence (if necessary), arrive at conclusions and modify these, if needed. Thereafter, the expert groups break up and all students return to their home teams. Each member of the home team now takes turns and shares with the team her understanding and insights and supports each others' learning.

Learning Together (LT): In this model, the students are divided into heterogeneous groups each of 4-5 members and each group prepares just one response sheet and each group receives recognition and praise according to the quality of the product. LT emphasises team-building, collaborative activities and promotes regular discussion among team members about how well their team is working and how their working can be made productive.

Group Investigation (GI): This is general plan of organisation. Under it, students work in small groups, use cooperative inquiry, group discussion, cooperative planning and projects. Students work in 2-6 member teams of their choice, they choose sub-units/sub-topics from the unit/topic being studied by the whole class and work on their chosen sub-units/sub-topics and prepare a group report. Each group presents its report to the class and thereby contributes to the learning of whole class.

9.3.26.2 Role of Teacher in CLA Classroom

Use of CLA in the classroom does not mean that the teacher is free to do whatever she likes.The teacher has to structure the tasks, set the goals and decide the sub-topics. She has to decide about

(i) How large will the groups be?

(ii) How will learners be selected for each group?

(iii) How much time will be devoted to group work and to whole class teaching?

(iv) What roles are needed for accomplishing group task and how are these to be assigned?

(v) What rewards, if any, will be provided for individual as well as group work?

Secondly, the teacher should (a) teach students how to communicate one's ideas and feelings, (b) show how to make their utterances complete and specific, (c) demonstrate how to make verbal and non-verbal messages congruent, (d) demonstrate and explain how to create an atmosphere of respect and trust in the group, (e) explain how to assess whether a message is properly received, (f) demonstrate how to paraphrase another's point of view, (g) illustrate how to negotiate meanings and understandings, (h) teach and show how to achieve participation and leadership.

Thirdly, the teacher, who uses CLA, should observe her students as they function in cooperative learning settings. She may have to intervene when necessary to help the students in achieving the team goal(s). Through monitoring, the teacher can come to know when a group will need assistance. Sometimes the teacher has to intervene when students appear to be at loss and do not know how to proceed further. Lastly, the teacher has to arrange for debriefing as well. During debriefing, the teacher may invite students to express how a group functioned and what factors retarded or accelerated group performance. Debriefing can motivate students to improve group performance.

For CLA, some tools, such as pre-test and post-test, cooperative study schedule and involvement in learning scale may be developed. Also some criteria may also be discussed for development of the problems. The development of the problem may be as follows:

While developing the problems, the investigator may be guided by Polya who remarks, '... a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development and misses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for and some means for independent thinking'. Hildebrand's scheme of categories of mathematical problems was used for developing problems. In this scheme, category-I problems use one or more simple mathematical principles and concepts; category-II problems may require some experimentation and collection of data before making students feel that a solution is possible, category-III problems concern situations which are complex versions of simple situations. These problems can be created by adding another dimension to the problem or making it such that it requires the study of a complete generalisation/abstraction before it can be solved; and category-IV problems lead to the formulation of general principles or the conjecturing and eventual proof of specific theorems. After this, findings may be analysed.

9.3.26.3 Implications

- Mixed ability grouping is better suited to effective learning of mathematics through CLA than homogeneous ability grouping
- Mathematics teachers normally lay greater emphasis on procedural knowledge than on conceptual knowledge while teaching mathematics. Effective use of CLA presupposes relatively greater conceptual knowledge on the part of the learners
- Mathematical problems are best presented as problematic so as to promote inquiry based learning of mathematics on the part of learners
- The language of mathematics is different from the language of other disciplines because of mathematical concepts it deals with. Terms used in mathematics are often

parasitic upon words used in every day communication. The teachers need to negotiate the meaning of these terms in mathematical context as part of the process of teaching. The learners should be enabled to derive mathematical procedures from verbally stated problems

- Peer interaction and learner-initiated discussion need to be encouraged by the mathematics teacher because these provide a window on the learner's mind. Learner interaction reveals how they approach and structure or restructure a problem
- Learning takes place more and faster in a non-threatening and goal oriented interactive environment. After all, knowledge is not in the brain, it is in the interaction. Vygotsky too emphasises that children learn better in a social constructivist environment
- The mere act of putting learners in cooperative learning situations does not work by itself. CLA is a tool in the hands of a teacher and the teacher needs to be trained in how to use it most effectively. Learners need to be provided feedback on how they can work more effectively and support each other's learning. The teacher's contribution in promoting effective learning in cooperative learning environment cannot be overemphasised. Her role is perhaps a lot more crucial here and she needs to devise innovative instructional strategies that can enable each learner to develop valid understanding and insight. To quote Romberg, 'In today's World, knowledge is seen as constructive', learning as occurring through active participation and teaching as guiding.

9.3.27 Stimulating Creativity and Inventiveness in Mathematics

Human beings are distinguished from other species by their special power of creativity and education, specially mathematics education. It is necessary to encourage and develop this inherent creativity in every child. Discovery - approach, pattern recognition approach, and freedom-to-learn approach are all essential for this.

9.3.27.1 Creativity and Education

If the destiny of a person is to become more and more creative, it is the function of Education to enable her/him to fulfill this destiny. The object of Education can no longer be just the transmission of knowledge already known, rather it should be transmission of the process of creation of new knowledge. The present knowledge has to be so communicated that the student learns the process of creating new knowledge as an essential by-product. In fact, even in the choice of the subjects for education, those subjects which are given a greater impetus to creativity should be given greater emphasis. By all standards, mathematics is one of the most important; if not the most important of such subjects, which if taught properly can teach the creative process at its best. This condition is important for mathematics as it is taught in present day schools; it very often dulls whatever innate creative capacity the child has. As such education for creativity must start right from the kindergarten stage and must be perservered throughout the education of the student.

9.3.27.2 Some Thoughts on Creativity in Mathematics Education

One of the best techniques of bringing about excellence in mathematics education is to awaken creativity in students. Similarly, only creative mathematicians can produce first-rate mathematics. The study of creativity is, therefore, of the greatest importance for mathematics education. Some of the thoughts are given below:

1. One of the purest forms of creative effort is in mathematics; it does not greatly depend on availability of equipment, nor on complexity of social phenomena. Creativity here is almost purely a human effort.
2. Mathematics can also be taught as a discipline in which intuition is used freely to make conjectures, hypotheses, generalisations and abstractions and in which logic enters only at the final stage to consolidate the gains in knowledge obtained by intuition. Children can be given the freedom to learn by *recreating* and *reinventing* mathematics.
3. Creative work in mathematics requires intuition, imagination, experimentation, judicious guessing, blundering, fumbling, hard work, tribulation and real thinking. Mathematics is the process of thinking rather than the final product. Knowing mathematics should imply using the mathematical habit of thinking.
4. Creative process is real mathematics.
5. At the lowest level, we have combinatorial creativity. There are thousands of ideas available to the mathematicians. He can make millions of premutations and combinations of ideas out of them.
6. Another type of creativity in mathematics is generalisational creativity. Here, given a set of concepts or theorems, we try to create a more general concept or theorem which includes the earlier ones as special cases. The search for comprehensive concepts and theorems provides an important motivation for creativity.
7. Another motive for creativity is abstraction. From concrete experiences of the outside World, we get abstract mathematical concepts of the first order. We may take up a number of abstract mathematical concepts of the first order and do another process of abstraction to get an abstract concept of the second order and so on. Each process of abstraction is an act of creativity and requires grouping, experimentation, deep involvement and taste.
8. For creative work in mathematics, we first guess theorems through experimentation and induction and then try to search for the axioms from which these can be deduced.

In creativity, we go from theorems to axioms, while in teaching we usually go from axioms to theorems. The didactic process should be as closely as possible parallel to the creative process. This may be difficult to achieve. However, the didactic process must not be just the opposite of the creative process.

9. Creativity implies flexibility and a willingness to accept ideas from everywhere. A problem in geometry may be solvable with the help of ideas from algebra or optics or mechanics or vectors or transformations. The axiomatic process restricts our freedom and so restricts creativity. Theorem- proving requires ingenuity, but problem-solving requires creativity.
10. The aim of mathematics is not just to solve problems, it is to solve significant problems. The selection of significant problems requires as much creativity as their solution.
11. Problems in physical, biological, social, management sciences, etc. motivate creative work in mathematics. Since these sciences grow at an exponential rate, challenges for mathematical creativity also increase at that rate.
12. Another motivation for creativity arises from the brain storming which occurs at seminars, conferences, discussions, etc.
13. For young students, mathematical olympiads provide a forum for exercising creativity.
14. Quite often, mathematical study of nature and society leads to stimulating ideas which are later studied for their own sake and lead to exciting creative work in mathematics. Equally often mathematical concepts created out of pure intellectual curiousity and mathematical taste lead to useful applications. This interaction between mathematics and its applications is vital for the right type of creativity in mathematics.
15. Problem-solving is the heart of creativity. We start sometimes from what is given, sometimes from what has to be proved or to be found and sometimes from both ends. In mathematics, we sometimes construct the fifth storey first on shaky foundations and then think of strengthening the foundations. The creative mind has to explore unknown territories and start probing at as many points as they appear promising. Creativity demands single-minded devotion.
16. A creative worker has a strong geometric intuition. He can visualise before his mind an exuberance of mathematical concepts moving at random, combining, separating and recombining and he just makes notes whenever he sees a beautiful pattern emerging. A creative worker really sees beauty bare.
17. Since mathematics is mostly a creation of the human mind, it can best be learnt by a child by recreating it himself.

18. Mathematics teaching may start from situations which are rich in mathematical possibilities and the students should be enabled to see these possibilities. Inspection, thinking, discarding, accepting, making errors and improvement have all to be involved.
19. Mathematical creativity expresses itself in generating new significant concepts, generalising a number of concepts or theorems, establishing connections between obviously unconnected concepts and theorems and establishing connections between facts of mathematics and facts of nature and society.
20. A direct attack on a mathematical problem is not always successful. The creative capacity consists in specialising the problem or even generalising it or thinking of related problems so that we ultimately approach the solution of the original problem.
21. Creativity in mathematics means creating beauty and order out of chaos and it should bring feelings of similar self-fulfilment as in music, painting and other great arts.
22. Mathematical creativity may be encouraged by the availability of suitable symbolism or may be hindered by its absence. In fact, creativity may express itself in designing the right symbolism for ideas.
23. Our mathematical heritage is too precious to be handed over to children on a plate. The creative process is an essential part of this heritage. If we offer mathematics as a finished product, we are offering the structure without its soul.
24. Let teaching creativity be an explicitly - stated goal of mathematical instruction. Even let this be the most important goal. But creative students can be taught only by creative teachers. We should, therefore, make all efforts to increase the number of creative teachers.

In view of the above thoughts, some methods may be adopted to stimulate the children to be creative and inventiveness:

9.3.27.3 Creativity in School Mathematics

1. Creative mathematics teachers have to take a greater interest in school mathematics. If school mathematics becomes more dynamic, more intellectual, more creative, it is bound to produce good research workers and these research workers will in their turn help in further rejuvenation of school mathematics.
2. The school teachers have to be trained in the new spirit. It is not only necessary to teach them the new concepts that are being introduced in schools, it is much now necessary to give them the spirit behind the new changes. It is a big problem. Changing fixed attitudes of minds is more difficult than teaching a few additional topics. Hard thinking and careful planning is required on the part of the teachers.

3. A great deal of literature will be necessary to express the means by which the various topics in both the previous syllabus and the new (present) syllabus can be presented creatively. At the elementary level, this will require preparation of creatively conceived audiovisual aids.
4. Gradually, the new textbooks will have to be characterised by the new approach.
5. Students have to be given the feeling that mathematics is a dynamic intellectual enterprise and a dynamic enterprise has to be creative. Free and divergent thinking is the essence of creativity. Students should be given freedom to explore mathematical concepts to create new connections among concepts to represent concepts in multiple ways to investigate problems and their solutions through non-standard means. This will allow them to become creative problem solver.
6. Pedagogical research should be encouraged. Every teacher has in his class a rich living material which enables her to make pedagogical experiments. These experiments should be discussed in the classroom, in seminars and with community of mathematics teachers and may be published in educational journals afterwards.

Thus, we can say that creativity is one of the greatest assets of a nation. One of the best means of encouraging mathematical creativity is through mathematical olympiads.

Mathematics educators and teachers will have to work collectively in developing tests of creativity which can be field tested and validated for future use. This is a challenging field for meaningful interdisciplinary cooperation.

9.3.27.4 Principles and Manipulational Skills

In mathematics, a similar problem is often posed as a problem of teaching principles is teaching manipulational skills. The present emphasis is undoubtedly on the later. We teach students certain algorithms for adding, multiplying, dividing, taking square roots, finding HCF, LCM, operations with decimals and fractions, etc. without telling them the 'why' of various algorithms. We give them sufficient practice and in many cases more than sufficient practice, in the use of the algorithms. After this practice, the student begins to feel these algorithms as natural and she begins to feel that mathematics is a science of algorithms (or bricks) only. It is true that mathematics is the science of algorithms, but it is the science which creates algorithms rather than the science which mechanically uses algorithms. Using algorithms always requires an intellectual activity of a much lower order than the activity of creation of algorithms. The nature of mathematics requires us to be more of creators than mechanical users. In fact, it is claimed that if principles are explained clearly and unambiguously, practice can be reduced at least by fifty percent for the same degree of skills to be achieved : it is, however, obvious that teaching principles is much more strenuous job than teaching skills through drilling since in order to teach principles one must have a good insight into the structures of mathematics and we must be able to communicate these to our students. To teach engineers is more difficult than to train mechanics and we require different types of teachers. At present we have enough persons to train mechanics of mathematics, what we need is teachers who can teach engineering of mathematics and teach it well.

EXERCISE 9.6

1. Which are central concepts in student team learning ?
2. What is the role of a mathematics teacher in cooperative learning approach (CLA) ?
3. List some requirements of creative work in mathematics.
4. Why is the study of creativity important for mathematics education ?

Summary

In this Unit, the following points have been discussed:

- Quality of education is influenced by the competence of teacher
- Strengths and weaknesses of learners
- Identification of strength and weaknesses by a competent teacher of learners
- For enriching mathematics learning, the following activities need to be undertaken:
  - Assisted learning consisting of peer and computer assisted learnings
  - Supplementary text materials consisting of exemplar problem books, teachers handbooks, reference books, enrichment materials and teaching aids, including mathematics laboratory.
  - Summer programmes and how to organise them
  - Correspondence courses and their methodology
  - Mathematics clubs, their need and importance, their activities and how to organise them
  - Contests and fairs
  - Laboratory approach of teaching-learning of mathematics
  - Recreational activities, such as number patterns, games, magic squares, riddles, puzzles, etc.
  - Cooperative learning consisting of its approach (CLA), role of a teacher in CLA classroom and its implications
  - Stimulating creativity and inventiveness in mathematics

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UNIT 10

PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

10.1 Introduction

Every subject has its own nature, characteristics and content related specifics. While talking about teaching and learning of an individual subject, these aspects become more pertinent. Due to a sense of fear and failure, disappointing curriculum, crude assessment, inadequate teacher preparation and insufficient support in teaching of mathematics (NCF–2005), the teaching of mathematics becomes more challenging and more demanding task. While dealing with abstractness of concepts using signs and symbols, simultaneously, the teacher has to try to make the learner visualise the concept in and around his/her own environment. For teaching and learning of mathematics, mathematics teachers should utilise not only classroom situations, but also situations and experiences from learner's locality. Though the role of the teacher has shifted with time from 'the sage on the stage' to 'the guide by the side,' if we go deep, it would be clear that the role of the teacher has become more challenging, more resourceful, more demanding, more updated, more sensitive, more concerning, more humanistic, more attentive, more facilitating and above of all more professional than ever before.

Learner's quest and his/her problems in learning of mathematics have changed over the years and have acquired new dimensions at present. This demands greater efficiency and more professionalism on the part of the teacher. Today's mathematics teacher needs to be highly skillful, up-to-date with enriched new knowledge, new curriculum and policies in mathematics and its pedagogy. Hence, professional development of mathematics teacher has become important and vital for better teaching and learning of mathematics. Professional development of Mathematics teachers can be achieved through Inservice Training (INSET) programmes, Mathematics Teachers' Associations, Journals, Conferences, Seminars and Workshops, etc. which promote professional growth of mathematics teachers. In this Unit, we shall discuss the role of mathematics teachers and different ways of their developing professional growth.

Learning Objectives

After studying this Unit, the student-teachers will be able to:

- develop the understanding of various types of inservice training programmes for mathematics teachers
- develop the understanding of role of mathematics teachers' associations, journals and other resource materials in mathematics education
- develop the understanding of various aspects pertaining to participation in conferences, seminars, webinars and workshops
- prepare own professional portfolio.
- become active researcher to plan, transact and modify various aspect of their teaching

10.2 Inservice Programme or Inservice Training Programme

Inservice training programmes, we mean the training programmes which are meant for inservice teachers. After joining the teaching service, some of the teachers may still be lacking in some necessary skills which were not the part of their pre-service teacher training programme or which were not gained due to lack of resources at that level. Without these skills, they may feel handicapped to cope up with the teaching-learning environment and classroom situations. Behari (2008), while talking about teacher education programmes, raises some basic questions which can help in identifying a high quality teacher at entry level:

(i) Does he/she have experience of teaching?

(ii) Does he/she have some generic skills of teaching?

(iii) Does he/she have some personal characteristics?

(iv) Does he/she know and understand his/her subject of teaching?

(v) Does he/she understand the educational function of his/her subject?

Many a times, the answer to a few of these questions is practically 'no'. If at least one of the answers is 'no', after entry in the teaching profession, there is a need to transform it into 'yes' through inservice teaching by attending to unattended needs. Some of the existing content, methodology and practices in teaching may require updation and enrichment. Sometimes, due to inclusion of some new concepts in curriculum and textbooks, methods of teaching are needed to be changed. For introducing changes in policies, curriculum and textbooks, it is essential to equip the mathematics teachers with desired understanding of these changes. It is further important to prepare them with the new strategies for implementation of these changes. Helsing, Howell, Kegan and Lahey (2008), while talking about professional development of educational leaders, have put forth some views which are equally useful for mathematics teachers. They stated, "for meaningful change in their school environments, demands often require a level of personal development, many adults may not yet have, there is a need for professional development programmes that are genuinely developmental."

> Explore the demands which require a certain level of professional development of mathematics teachers. Discuss with your peers about level of such development and factors affecting this level.

All these can be achieved through inservice training programmes for teachers. Inservice programmes can play a significant role in the professional growth of teachers and also function as an agent for change in school related problems. These further can help teacher in gaining the confidence by engaging them professionally. These programmes may be organised through face-to-face mode as well as teleconferencing mode. Ramadas (2010) focuses on the use of teleconferencing in inservice training programmes for teacher trainers, which can be seen as a good directive for INSET programmes for mathematics teachers too. He concludes, "The two way audio-video interactive technologies used in the teleconferencing, helped not only to train a large number of target participants, but also brought about positive attitudinal change in them." It can be as advantageous as discussion on a single platform and on a real time basis. Further, time constraint in these programmes can be overcome by using e-mail technology and telephonic contacts.

> Explore the reports and recommendations of the Education Commission (popularly known as Kothari Commission) (1964-66), the National Commission on Teachers (1983-85), the Acharya Ramamurthi Review Committee (1990) to know more about inservice education.

The inservice programme can deal with the following:

- Motivation for learning the topics in mathematics
- Application of school mathematics

- Strategies for involvement of students physically, emotionally and intellectually in the learning process of mathematics
- Use of mathematics laboratory
- Knowledge of mathematics competitions like Mathematical Olympiad
- Exploring alternative methods of solutions and proofs
- Discussion on various problems of teaching and learning in mathematics and sharing the classroom experiences
- Discussion on historical references and anecdotes in mathematics and their use in teaching-learning of mathematics
- Projects in mathematics
- Strategies for problem-posing and problem-solving as foci of mathematical teaching
- Discussion on mathematics curriculum, including mathematics books
- Various evaluative techniques in mathematics.

National Curriculum Framework for Teacher Education (NCFTE–2009), brought out by National Council for Teacher Education (NCTE), gives the broad aims of continuing professional development programmes for teachers as follows:

- to explore, reflect on and develop one's own practice
- to deepen one's knowledge and update oneself about one's academic discipline or other areas of school curriculum
- to research and reflect on learners and their education
- to understand and update oneself on educational and social issues
- to prepare for other roles professionally linked to education/teaching, such as teacher education, curriculum development or counselling
- to break out of intellectual isolation and share experiences and insights with others in the field, both teachers and academicians working in the area of specific disciplines as well as intellectuals in the immediate and wider society.

For a variety of aims and objectives, various types of inservice training programmes are required.

10.2.1 Types of Inservice Programme for Mathematics Teachers

The inservice training programmes for mathematics teachers can be categorised into three types:

10.2.1.1. Skill enhancement programme

10.2.1.2. Updation and enrichment programme

10.2.1.3. Introductory programme for new policy and the curriculum.

10.2.1.1 Skill Enhancement Programme

There are some skills which were not required earlier for teaching of mathematics. For instance, a few years back, the knowledge of computers and skills required to operate computer were not felt as essential. But, now a days, the knowledge of computers and skills like developing powerpoint presentations etc. are thought to be very important for all teachers in general, mathematics teachers in particular. Animation and graphics in computers can be used to develop certain concepts in school mathematics using already available mathematical educational softwares. Computer softwares also generally help in drill exercises needed for consolidation of learning. Most of the students feel comfortable and are becoming more habitual to submitting their assignments and projects through computer texts and presentations. If a teacher is not equipped with the skills of using and evaluating these, how can she/he do justice to the students' work. So, skill enhancement programmes are very important in this regard.

For example 'Intel Teach to Future Programme' which is of about twenty one days duration for teachers of Directorate of Education, Government of NCT of Delhi was organised in Delhi in 2003 is a skill enhancement programme. In this programme, some selective teachers were given exhaustive training for developing computer skills pertaining to MS Word, MS Powerpoint, MS Excel, Internet etc. through hands on experience, discussion, group collaboration and project work. These teachers were nominated as 'Master Trainers' and they in turn were supposed to train ten teachers each at their concerned schools after completion of the programme.

> Is this programme, under private sector or public sector? Explore further for more skill enhancement programmes.

10.2.1.2 Updation and Enrichment Programme

Updation and enrichment in existing practices and content is an important factor for better teaching-learning of mathematics on the part of mathematics teachers. There are inservice training programmes meant for this purpose. These programmes are mainly organised by SCERTs of respective states. These programmes are of various durations which are varying from time to time. Most well known programmes are of twenty one days.

As a note for acquaintance of development in these programmes, earlier the applications from the interested teachers for attending these programmes were invited and only interested or needy teachers used to go through these programmes. Later on, it was being felt that such programmes are equally important for all the teachers. So, in some of the states, it has now become compulsory for almost all the teachers to attend these programmes on annual or biannual basis for a certain number of days.

These programmes may be spread over to two or more sessions to evolve better outcome on the part of school system and better participation of teachers during vacations of schools, so that the normal functioning of schools will not be affected.

> Explore for updation and enrichment programmes held in various states of India. Are there some programmes conducted by private sector too? If yes, explore and compare these programmes.

10.2.1.3 Introductory Programme for New Policy and the Curriculum

Whenever a new education policy or new textbooks are introduced, there is a need for inservice teachers to be trained in the policies brought, changes in the textbooks and use of the changed curriculum and textbooks. Moreover teachers may be given participatory role in the feedback for these new policies, curriculum and textbooks.

New textbooks were developed by NCERT based on NCF–2005. A number of orientation programmes for 'master trainers' were held in 2006 onwards to introduce new textbooks based on the guidelines as envisaged in NCF–2005 in all subjects in general and mathematics textbooks in particular across the country. Few of these exclusive programmes were organised through teleconferencing also.

A series of training programmes were also organised in past by NCERT, whenever new instructional materials were developed under different frameworks/policy.

As another instance, Government of NCT of Delhi introduced a set of new textbooks for Classes VI, VII and VIII in mathematics for the schools under Directorate of Education. An introductory training programme namely 'Indra Dhanush' was organised by Government of Delhi for teachers to get acquainted with theory and methodology for teaching through these textbooks.

> Can you explore for more such programmes held in other states too?

10.3 Role of Mathematics Teachers' Association

Mathematics teachers are active not only for their own and subjective growth, but also endeavouring for professional development of mathematics teachers' community. They have established associations of mathematics teachers. These associations are sometimes formal and sometimes non-formal. These associations have their own established norms, guidelines,

lines of actions, plan for activities and their own set up. These sometimes have their own office bearers and executives too. These office bearers, plan and act for the professional development of mathematics teachers with the cooperation of their fellow members and concerned stake holders.

A mathematics teachers' association can bestow opportunities for professional growth as revealed from various sources to teachers:

- It can provide a common platform for teachers where they can share their experiences time-to-time and discuss mathematics curriculum, instruction and assessment techniques
- It can provide opportunities to meet or invite prominent mathematicians/ educationists at national/international level on a regular basis to share their experiences
- It can initiate communication with members and other groups or organisations having an interest in mathematics education
- It can bring out a mathematics journal
- It can establish a website for communication among mathematics teachers in the region
- It can suggest guidelines for developing new syllabus and textbooks on mathematics on the basis of experiences of teachers to the concerned authorities.
- It can discuss participation in national and international meetings related to leadership in mathematics education
- It can organise orientation programmes for their member-teachers with the help of some other organisations.

Some well known associations in this category are - National Council of Teachers of Mathematics (NCTM), National Centre for Excellence in the Teaching of Mathematics (NCETM), National Education Association (NEA), International Congress of Mathematicians (ICM), International Mathematical Union (IMU), American Federation of Teachers (AFT), etc. (may be explored further from the web and other resources). Ollerton (2006) mentions two such associations, namely Association of Teachers of Mathematics (ATM) and Mathematical Association (MA). These have their own websites. They update it time to time. They provide a list of activities, which they plan and hold for mathematics teachers on the website.

There are certain mathematics teachers' associations in India too viz. Delhi Association of Mathematics Teachers (DAMT), Association of Mathematics Teachers of India (AMTI), Kerala Mathematics Teacher's Association (KMTA), etc.

Can you explore some more national and international associations of mathematics teachers?

Apart from these associations, there are other organisations too which are working directly or indirectly in the field of mathematics education. Commonwealth of Learning (COL) is an intergovernmental organisation created by commonwealth heads of government to encourage the development and sharing of open learning and distance education knowledge, resources and technologies' learning for development. It is working in the field of all areas of education, including mathematics education.

Some research institutes in mathematics are - Tata Institute of Fundamental Research (TIFR), Mumbai; Institute of Mathematical Sciences (IMSc), Chennai; Indian Statistical Institute (ISI), Delhi; the Harish Chandra Research Institute (HRI), Allahabad; Indian Institute of Science (IISc), Banglore; Chennai Mathematical Institute (CMI), Chennai; Indian Institute of Science Education and Research (IISER) Pune, etc.

10.4 Role of Journals

A journal can be understood as a periodical dedicated to a particular subject or a magazine containing articles relating to a particular subject or profession.

Journal in any academic field is the mirror of the current and most recent research activities in that field to give to all other concerned persons the acquaintance of the progress in the field. This is the most powerful tool to give directions for further research at mass level. This gives a name, fame and recognition to not only the researcher, but also to a new line of action, area, methodology and ideology in the academic field. There are journals at regional, national and international levels.

Since the mathematics and its teaching has been a crucial centre of concern for educationists for a long time, there were consistent novel efforts in this field. Journals have spread these efforts and the recognition of outputs to a mass level. Hence, journal has been evolved as a powerful instrument for the professional development of mathematics teachers.

These journals motivate mathematics teachers to share their experiences with other readers and give ideas that help teachers to solve their own classroom problems. Sometimes, these journals give a new ideology to cope with the community and the working environment of mathematics teachers. A remarkable advantage, now a days, of these journals is that many journals are available not only as print journals, but also as online journals and open journals. It also cannot be denied that every mathematics teacher struggles hard in facilitating the learning of his/her students. While undergoing this process, he/she applies various novel strategies, content, methodology; self prepared text material, self made tests etc. to find the solution of problems arising to achieve the goal of better mathematics learning to each and every student. Hence, knowingly or unknowingly she/he comes up with some new ideas or novel solutions to the situation in order to reach to the heart of their students and empathising crux of their problems. This output is also a sort of research output for a mathematics teacher. They should try to get it published in the journals. Hence, every mathematics teacher should contribute papers and articles in journals in order to reach the academic heart of their students and in turn for the professional development of the entire mathematics teachers' community, including himself/herself. Few journals/magazines are: *The Mathematics in School, The School Science, The Mathematical Gazette, The Junior Mathematician, The Mathematics Teacher, The Mathematics Education, The Mathematical Intelligencer, The Teaching Children Mathematics, The Primary Teacher, The Journal for Research in Mathematics Education, etc.*

Can you explore some more national and international journals?

10.4.1 How to Publish a Paper in a Journal?

Different journals have their own format for publishing research papers and research notes. Some of the journals have space for book reviews also. We, here, are more concerned about research papers. Once a mathematics teacher conducts a research, may be fundamental research or applied research or action research, he/she should prepare a detailed write up of the research work and arrange this write up (sometimes called manuscript) in the required format as directed by the particular journal in which researcher wishes to get published her/his research paper. A suggestive format may be given as: the title page, the abstract, introduction, significance, delimitations, sampling, tools, procedure for data collection, output, analysis and discussion, conclusion and references. This is an important aspect that almost all journals ask for references, not the bibliography. As accepted by wide number of researchers, references contain note of only those documents or websites which are actually mentioned in the text of the paper or research report while bibliography can contain those too which are read and used, but not actually mentioned in the text of the paper or research report. Thereafter, if they wish they may consult experts for feedback and suggestions. After incorporating the rational feedback and suggestions, the research paper is ready to be sent to the editor of the journal for publication. Mathematics teacher can send this paper to the editor and should wait to get the response and feedback again from the editors. After incorporating these suggestions, following instructions from the journal authority and sending back the paper to the editor, the paper can be hoped to be published in the journal. When getting the paper published and receiving the copy, one should not forget to share the paper with every concerned person pertaining to the field of mathematics and mathematics teaching in her/his community for the benefit of all at mass level.

10.5 Role of other Resource Material in Mathematics Education

Clements and Ellerton (1996) state, "Mathematics Education is concerned with the development and implementation of appropriate mathematics curricula, and with all issues associated with the teaching and learning of mathematics. In keeping with the concept of lifelong learning, mathematics education covers learners of all ages and at all levels from early childhood to adult." It is obvious, here, that mathematics education is not just confined to curricula, classrooms, teachers and learners in school, but also other issues like resources in mathematics. A detailed discussion about various resource materials for mathematics education and learning of mathematics has been dealt in Unit 7 of this book. There are other resource materials at higher levels too, viz., regional, national and international level. Roblyer and Dickey (2008) depicts technology integration strategies for mathematics with sample resources and activities:

Table 10.1 Technology Integration Strategies for Mathematics and Sample Resources

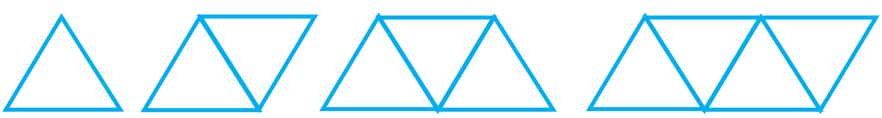
S.No.	Technology Integration Strategies	Benefits	Sample Resources and Activities
1.	Using virtual manipulatives	• Supports hands on activities for learning mathematics • Offers flexible environments for exploring complex concepts • Provides a concrete representation of abstract concepts	• http://matti.usu.edu/nlvm/index.html • http://www.shodor.org· • http://www.ies.co.jp/math/java/index.html • http://www.mste.uiuc.edu/java/default.php
2.	Fostering mathematical problem-solving	• Helps students gather data to use in problem-solving • Provides rich, motivating problem-solving environments	• Calculator Based Laboratories • Software (e.g., Riverdeep's Zoombinis, Thinkin' Things)·

		• Gives students opportunities to apply mathematical knowledge and skills in meaningful contexts	• Programming languages
3.	Allowing representation of mathematical principles	• Makes abstract mathematical concepts more visual and easier to understand • Gives students an environment in which to make discoveries and conjectures related to geometry concepts and objects • Provides easy access to many data sets	• Graphing calculators • Software (e.g., Geometers Sketchpad, KaleidoMania) • Spreadsheets
4.	Implementing data-curriculum	• Provides real statistics to support investigations that are timely and relevant • Supports development of student's knowledge and skill related to data analysis • Allows exploration and presentation of data in a graphical form	• Software (e.g., Fathom, Tabletop, Tabletop Jr.) • http://www.census.gov • Spreadsheets • Statistical software (e.g., StatCrunch, http://www.statcrunch.com/)
5.	Supporting mathematics related communications	• Allows easy contacts with experts • Promotes social interaction and discourse about mathematics • Allows teachers to reach other teachers for the exchange of ideas	• http://mathforum.org/dr.math/ • http://mathforum.org/pow/ http://mathforum.org/discussions/ • http://www.nctm.org/onmath

6.	Motivating skill building and practice	• Provides motivating practice in foundation skills needed for higher order learning • Provides guided instruction within a structured learning environment • Delivers instruction when teacher may not be available	• http://www.boxermath.com • http://www.plato.com • http://www.pearsondigital.com

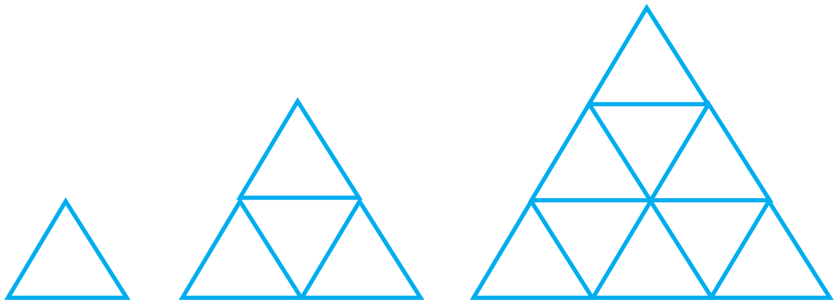
These resources and resource materials can play a vital role in the professional development and professional growth of mathematics teachers, and hence, the field of mathematics education. Simmons and Hawkins (2010), in this context, talk of some of the technology changes like virtual learning environments, e-portfolios, interactive white boards, electronic voting systems, social networking and innovational use of technology, e.g., future lab, the knowledge of which is just like must for teachers. These are important resources for education. Explore more about these technological resources and their applications in the field of mathematics education. Even activities can be used not only as a way, but also as some good resources in mathematics education. A translated book 'Ganit ki Gatividhiyaan' by Gupta (2002) depicts so many activities in mathematics teaching-learning, which are really helpful in mathematics education. It throws light not only on different strategies for teaching of mathematics, but also about resources and teaching materials, which can be easily available in the locality of the learner, for instance, use of matchsticks for linear and quadratic patterns. Here, linear can be understood as $a + (n - 1)\,d$ which also denotes general term of an Arithmetic Progression and quadratic means an expression involving a highest degree term n^2, i.e., $n \times n$ (see Fig.10.1).

See what is the relationship in number of triangles formed in each successive pattern. It is another issue of concern, how many number of matchsticks are being required in forming successive pattern. What is the relationship between number of triangles formed and number of matchsticks required? If in Fig 10.1 (i), number of triangles formed are $a + (n - 1)\,d$, what will be the values of a, n and d in all successive patterns? In Fig 10.1 (ii), if number of triangles formed are represented by $n \times n$, find values of n for all successive patterns.



(i) Linear patterns

Fig. 10.1



(ii) Quadratic patterns

Fig. 10.1

ATM MATs is a resource created by Adrien Pinel, which provides learners with a practical experience to create and explore 3-D solids and 2-D shapes alongwith kinesthetic learning. Ollerton (2006) found ATM MAT as a very interesting resource for making students learn a very central aspect of mathematics, i.e., exploring structures.

10.6 Professional Growth

There are other means also other than journals for professional growth of mathematics teachers at higher level, viz., regional, national and international level. There are conferences, seminars, workshops and competitions at these higher levels which encourage the professional development of mathematics teachers and in turn facilitate the learning of mathematics on the part of their students. For instance, Department of Teacher Education and Extension of NCERT holds an 'All India Competition on Innovative Practices and Experiments in Education for Schools and Teacher Educational Institutions' at national level. Mathematics teachers can play a vital role in such competitions for their professional growth. Let us discuss in detail now.

10.6.1 Participation in Conferences

A conference can be understood as a prearranged meeting for consultation or exchange of information. It can also be seen as a discussion among participants on a particular theme or topic and confer about the topic.

Mathematics teachers can participate in conferences, which will add to their professional growth. If they participate in conferences, it will help them in two ways. Firstly, they can share their experiences, with others and secondly they can clear their individual doubts arose during their teaching-learning of mathematics. There are conferences at regional, national and international level in which mathematics teachers can participate.

10.6.1.1 How to Participate in a Conference

Organisation which holds the conference announces the theme, subthemes, format, eligibility, important dates, how to apply, contribution, etc. for information of all. Any mathematics teacher agreeing with the norms for the conference can send request to the concerned authority to participate in the conference. When accepted, teachers can follow the rest of the guidelines, conveyed time to time to them, and participate in the conference.

10.6.2 Participation in Seminars

Seminar is a sort of face to face acquainting the people about one's own research and sometimes giving a space for discussion too. This is generally organised at a mass level. Mathematics teachers can participate in seminars for their professional growth and hence, adding to their professional portfolio.

10.6.2.1 How to Participate in a Seminar

Like the conferences, the organisation which holds the seminar announces the theme, subthemes, format, eligibility, important dates, how to apply, contribution, etc. for the information of all. Any mathematics teacher, agreeing with the norms for the seminar can send request to the concerned authority to participate in the seminar. When accepted, the teachers can follow the rest of the guidelines conveyed time to time to them, and participate in the seminar.

10.6.3 Participation in Webinars

Due to some of the major delimitations of place, distance, travel, officially allotted assignments within the premises and limited number of seats for participants; most of the professionals are not able to participate in the seminars. Technology tried to impart a solution to this problem. And hence, the concept of e-presentations and webinar evolved. The term 'webinar' has been originated from 'web' and 'seminar,' which clearly indicates about the use of web resources for the seminar. There are some organisations, for instance the Open Education Resource Foundation (OERF), which are using a new version of audio-video teleconferencing with the help of some other latest technologies for enabling most of the interested people at mass level to attend online seminars removing the constraints of place, time, travel and the distance. For attending or participating in webinar, one should have web or internet facilities at his/her place.

10.6.3.1 How to Participate in a Webinar

Except the use of web, it is almost same as that of seminar. Organisation, which holds the webinar announces the theme, subthemes, format, eligibility, important dates, how to apply and make contribution etc. for the information of all. Any mathematics teacher, agreeing with the norms for the webinar can send request to the concerned authority to participate in the seminar. When accepted, teachers can follow the rest of the guidelines, conveyed time to time to them, and participate in the webinar.

10.6.4 Participation in Workshops

Fig. 10.2

Workshops are more oriented towards productive outcomes. These provide one sort of real outcomes then and there, where the workshop is being organised. It may hold small duration activities, projects, presentations, etc. For instance, a three day 'workshop for redistribution of existing annual curriculum into semester system' may result into a semesterised curriculum. A workshop has predetermined specific objectives to be achieved.

10.6.4.1 How to Participate in a Workshop

It can be held at local as well as higher organising level. Organisation, which holds the workshop announces the theme, subthemes, format, eligibility, important dates, how to apply and make contribution etc. for the information of all. Any mathematics teacher, agreeing with the norms for the workshop can send request to the concerned authority to participate in the workshop. When accepted, teachers can follow the rest of the guidelines, conveyed time to time to them, and participate in the workshop. However, in some of the workshops, the organisers themselves select and invite the prospective participants, keeping in mind their specific needs.

10.6.5 Professional Portfolio

One of the best means discussed by Beckmann, Thompson and Rubenstein (2010), for overall expression and monitoring of professional growth of mathematics teachers may be in the form of 'Professional Portfolio'. A professional portfolio includes evidence of expertise related to the profession of an individual. It can reveal individual views about understanding of one's role as a teacher and as a professional, one's continued professional development and professional progress of the individual teacher for providing not only a tool for interview, case for advancement in the career, but also as a consequence, for the betterment of the field of mathematics education. A suggestive professional portfolio can be prepared as consisting of following points:

- Mission statement
- Professional resume
  - Education
  - Professional work experience
  - Related experience
  - Grants and awards
  - Publications
  - Presentations
  - Professional development experiences
  - Professional affiliations
  - Hobbies and interests
- Personal statement
- Teaching statement
- Awards
- Letters of support
- Examples of exemplary teaching materials
- Examples of assessments and student's work
- Pictures of your working with students
- Pictures of exemplary classroom displays you have created
- Professional presentations
- Published papers
- Grant proposals

Though most of these above mentioned points are self explanatory, but student-teachers are advised to explore more on these points by discussing with their peers, colleagues, school mentors, subject experts and professional career experts.

EXERCISE 10.1

1. Critically analyse the role of inservice programme for mathematics teachers. Elaborate your answer.
2. How associating with a mathematics teachers' association can help the mathematics teacher in his/her professional growth? What are the possible ways to establish a mathematics teachers' association at a small scale?
3. Discuss various steps for preparing a research paper for a journal in mathematics education in detail by taking an appropriate example.
4. Explore the availability and use of other learning resources for mathematics which can help you handling a mathematics classroom situation. Illustrate with examples.
5. As a mathematics teacher, what steps would you take for preparing yourself to participate in a conference?
6. Suggest some of the activities you would like to perform in a workshop "Development of projects in Mathematics at secondary stage".
7. Critically examine the role of professional portfolio in the professional growth of mathematics teachers. What sort of problems you may face in preparing a good professional portfolio? How can you overcome these problems?

Summary

Inservice programmes or inservice training programmes play a vital role in enhancing skills, introduction of new policy and curriculum, updation and enrichment for mathematics teachers for the betterment of the field of mathematics education. These programmes can be of three types on the basis of their purposes - (i) Skill enhancement programme, (ii) Updation and enrichment programmes, and (iii) Introductory programme for new policy and the curriculum. NCERT, SCERTs and some other organisations organise these programmes for teachers in India. For professional growth of mathematics teachers, mathematics teachers' associations, journals, conferences, seminars, webinars and workshops are some of the very important means which can help mathematics teachers to enhance qualitatively as well as quantitatively their professional portfolio. The ultimate aim for such programmes is to develop community of passionate and commited mathematics teachers to have capacity to grow professionally.

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Mathematics is more than a technique. Learning mathematics is acquiring an attitude of mathematical behaviour. Mathematicians are inclined to teach mathematics as an aim itself. However, it should be taught with a view to its educational consequences. Mathematics should not be taught to fit a minority but to fit everybody and students should learn not only mathematics, but what to do with it.

H. Freudenthal

NOTE

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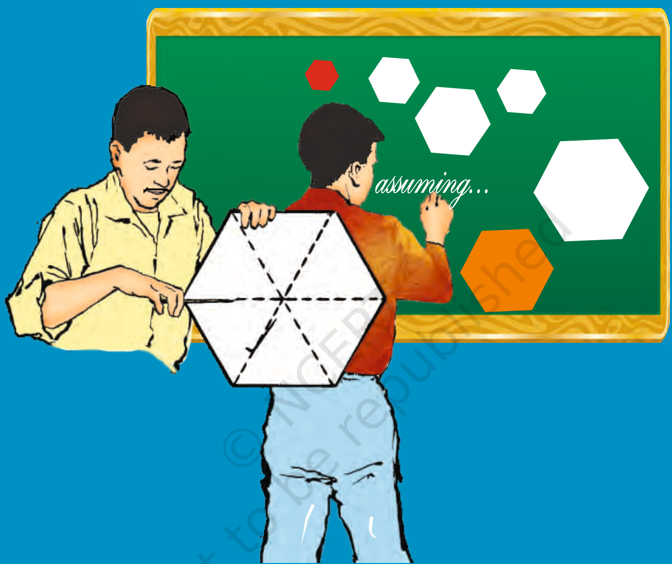

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विद्यया ऽमृतमश्नुते

एन सी ई आर टी
NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

ISBN 978-93-5007-189-2